Improving Dynamic Programming

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Optimal cost of larger items is based on optimal cost of smaller items.

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Chain Matrix Multiplication: Finding "cheapest" way to multiply matrices $A_1, \ldots, A_n$ where A_i is a $p_{i-1} \times p_i$ matrix.

$$m[i,j] = \begin{cases} 0 & \text{if } i = j \\ \min_{i \le k < j} m[i,k] + m[k+1,j] + p_{i-1} p_k p_j & \text{if } i > j \end{cases}$$

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Want $m[1,n]$ and corresponding set of multiplications

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Longest Common Subsequence: Find LCS of strings $X =< x, 1, \ldots, x_m >$, $Y =< y_1, \ldots, y_n >$.

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1,j-1]+1 & \text{if } i,j > 0 \text{ and } x_i = x_j \\ \max(c[i-1,j],\ c[i,j-1]) & \text{if } i,j > 0 \text{ and } x_i \neq x_j \end{cases}$$

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Individual techniques found wide applicability in areas outside original application.

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New speedups are still being found, still on ad-hoc basis.
Crying need for a general theory of speedups, that can be referenced by application users.

In this talk, will combine

- one well-known time speedup:
  Monge Property + SMAWK algorithm and
- one basic $\Theta(n)$ space improvement
  (Hirschberg 1975)

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Calculating $H(n, D)$ requires only $O(n)$ space.

Note that storing the table uses $\Theta(Dn)$ space,
where D could be quite large.

Naive method of constructing solution from DP table,
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Will see how to reduce this to $O(n)$ space.

Outline

- The Monge Speedup

- Saving Space While Saving Time

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7	2	4	3	9	9
5	1	5	1	6	5
7	1	2	0	3	1
9	4	5	1	3	2
8	4	5	3	4	3
9	6	7	5	6	5

$\text{RM}_M(1) = 2$

$\text{RM}_M(2) = 4$

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$\text{RM}_M(5) = 6$

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- 2×2 monotone matrices have form

2	4
4	5

2	3
5	3

7	1
2	2

~~7~~	~~1~~
~~2~~	~~3~~

- An $m \times n$ matrix M is Totally Monotone (TM) if every 2×2 submatrix is Monotone.

(submatrix: not necessarily contiguous in the original matrix)

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- SMAWK Algorithm
  [Aggarwal, Klawe, Moran, Shor, Wilber (1986)]
  - If M is Totally Monotone,
    all m row minima can be found in $O(m+n)$ time.
  - Usually $m = \Theta(n)$
    $\Theta(n)$ speedup: $O(n^2)$ down to $O(n)$.
  - See http://www.cs.ust.hk/mjg_lib/Classes/COMP572_Fall07/Notes/SMAWK.pdf for proof

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- SMAWK was culmination of decade(s) of work on similar problems; speedups using convexity and concavity.
  Has been used to speed up many DP problems, e.g., computational geometry, bioinformatics, k-center on a line, etc.

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- Also, if $\forall i, j$, $\quad M_{i,j} + M_{i+1,j+1} \leq M_{i+1,j} + M_{i,j+1}$,
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  $\Rightarrow$ M is Monge.
- $\Rightarrow$ Only need to prove Monge property for adjacent rows and columns.

An Example of a Monge Matrix

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From http://en.wikipedia.org/wiki/Monge_array

$$\begin{bmatrix} 10 & 17 & 13 & 28 & 23 \\ 17 & 22 & 16 & 29 & 23 \\ 24 & 28 & 22 & 34 & 24 \\ 11 & 13 & 6 & 17 & 7 \\ 45 & 44 & 32 & 37 & 23 \\ 36 & 33 & 19 & 21 & 6 \\ 75 & 66 & 51 & 53 & 34 \end{bmatrix}$$

To see that it's Monge, only need to check the 24 instances of

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Monge (or Total Monotonicity) seems an esoteric condition. In reality, it occurs *very* often.

Finding row minima can be used as a DP primitive.
$\Rightarrow$ the SMAWK algorithm can be used to speed up many DPs.

Using The Monge Property

Suppose we are given DP ($H(i,0)$ known, $i \leq n$, $d \leq D$):

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So, $O(Dn)$ time to calculate $H(n,d)$ and we are done!

Examples of

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$i \le n,\ d \le D$

- Length Limited Huffman Codes $0 \le p_1 \le p_2 \le \cdots \le p_n$

  $w(j,i) = S_{2j-i}$ where $S_k = \sum_{i=1}^{k} p_i$.

  $H(n-1, D)$ is cost of min-cost D-limited code

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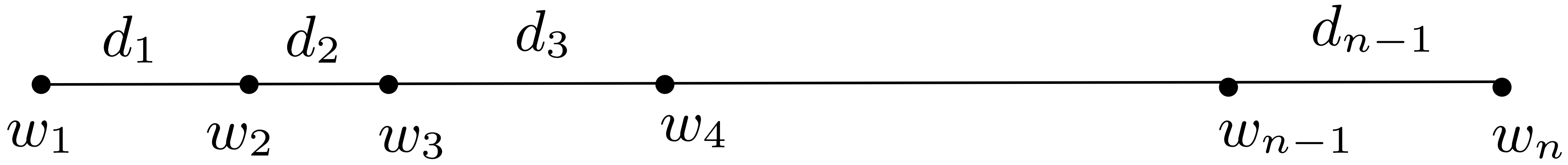
- Wireless mobile paging $\quad p_1 \ge p_2 \ge \cdots \ge p_n \ge 0$

  $w(j,i) = i\left(\sum_{\ell=j+1}^{i} p_\ell\right)$

  $H(n, D)$ is min expected bandwidth required to page all items using $\le D$ paging rounds

- D-Medians on a Directed Line *Woeginger '00*



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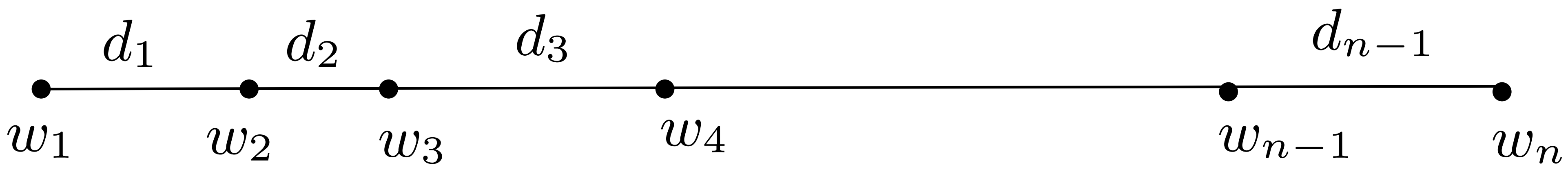

Identify D nodes as service centers.

Nodes can only be serviced by node to their left (or themselves) so node 1 must be a service center.

Cost of servicing request w_i, is w_i times distance from node i to nearest service center.

Problem is to find location of D service centers
that minimize total service cost.

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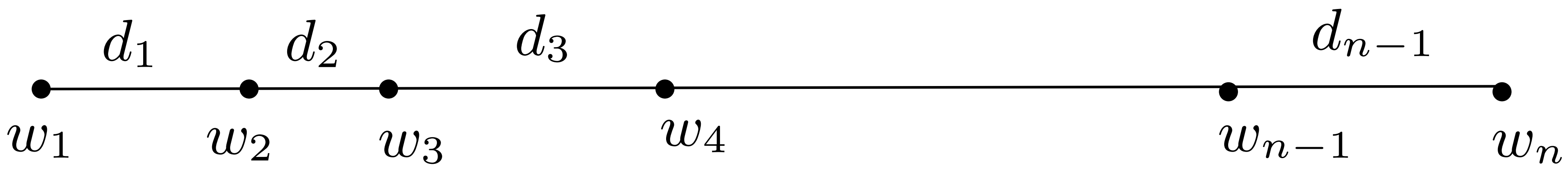

Let $H(i,d)$ be cost of servicing nodes $[1,i]$ using exactly d servers.

$$H(i,d) = \begin{cases} 0 & n = d \\ w(0,i) & d = 0,\ i \geq 1 \\ \min\limits_{d-1 \leq j < i} (H(j,d-1) + w(j,i)), & 1 \leq d < n \end{cases}$$

$$w(j,i) = \sum_{l=j+1}^{i} w_l(v_l - v_{j+1}), \quad v_k = \sum_{j=1}^{k-1} d_j$$

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- D-Medians on a Directed Line $w(j,i) = \sum_{l=j+1}^{i} w_l (v_l - v_{j+1})$

All these $w(j,i) = w_{j,i}$ satisfy Monge property

$$w_{j,i} + w_{j+1,i+1} \le w_{j,i+1} + w_{j+1,i}$$

$\Rightarrow H(n,D)$ can be calculated in $O(nD)$ time

Outline

- Review of the Monge Speedup

- Saving Space While Saving Time

Given a DP in the form

$$H(i,d) = \min_{0\le j<i}\Big(H(j,d-1)+w(j,i)\Big) \qquad \begin{matrix} 0\le i\le n \\ 0\le d\le D \end{matrix}$$

in which, the $w(j,i)$ are Monge, e.g., D-limited Huffman Encoding, D-Median on a line or Wireless Paging , the $H(\cdot,\cdot)$ table can be filled in using only $O(nD)$ time.

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Furthermore, calculation of $H(\cdot,d)$ only requires knowledge of $H(\cdot,d-1)$. So, if $H(n,D)$ is final goal, we can fill in table iteratively, for $d = 1,2,\ldots,D$, using only $O(n)$ space.

On the other hand, finding actual "solution path" of DP, corresponding to min-cost tree, median locations or paging schedule, requires backtracking through DP table. This implies storing entire table, using $\Theta(nD)$ space.

Context:

$$H(i,d) = \min_{0 \le j < i} \Big(H(j, d-1) + w(j,i) \Big) \qquad \begin{array}{l} 0 \le i \le n \\ 0 \le d \le D \end{array}$$

D-Length-Limited Huffman Coding

(*) $w(j,i) = S_{2j-i}$ where $S_k = \sum_{i=1}^{k} p_i$.

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Would like to reduce space for (*) down to $\Theta(n)$

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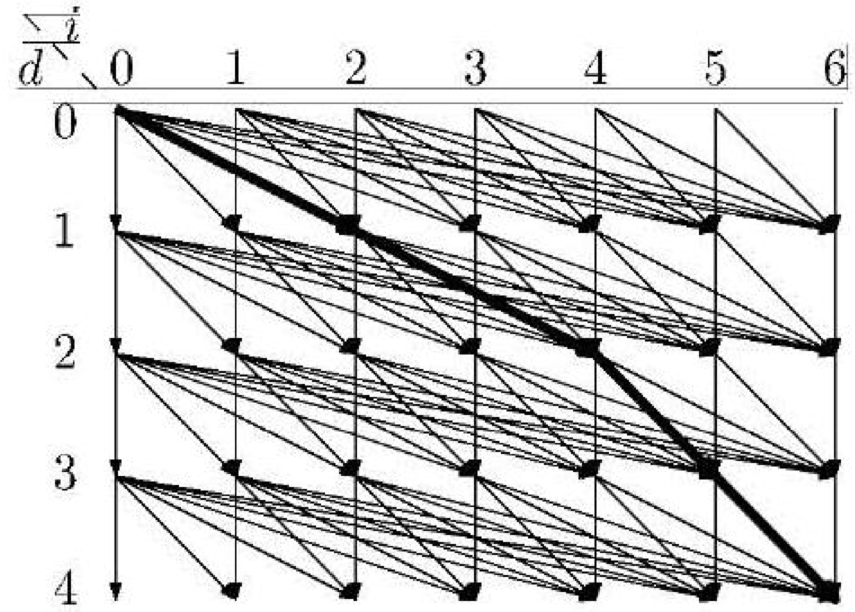
Alternative Interpretation:

Consider a layered graph in which edges only go down one level and to the right.



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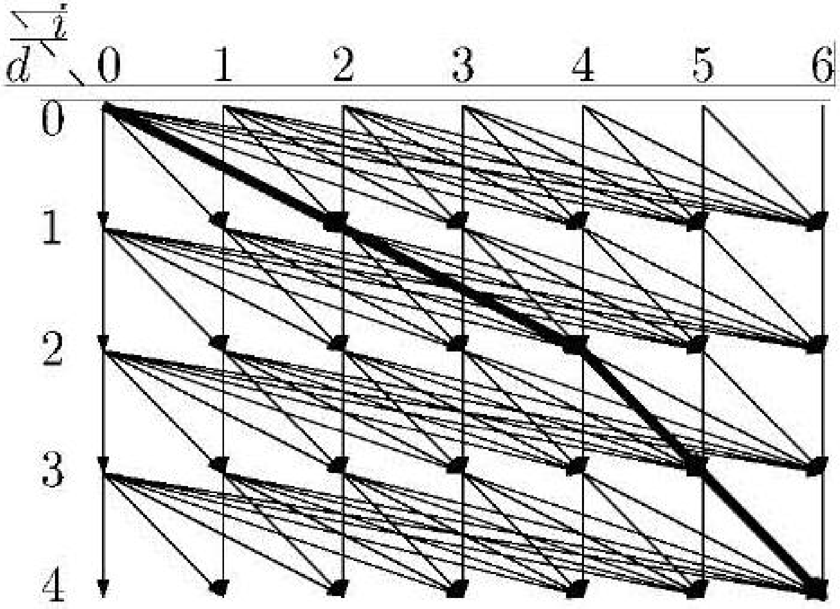

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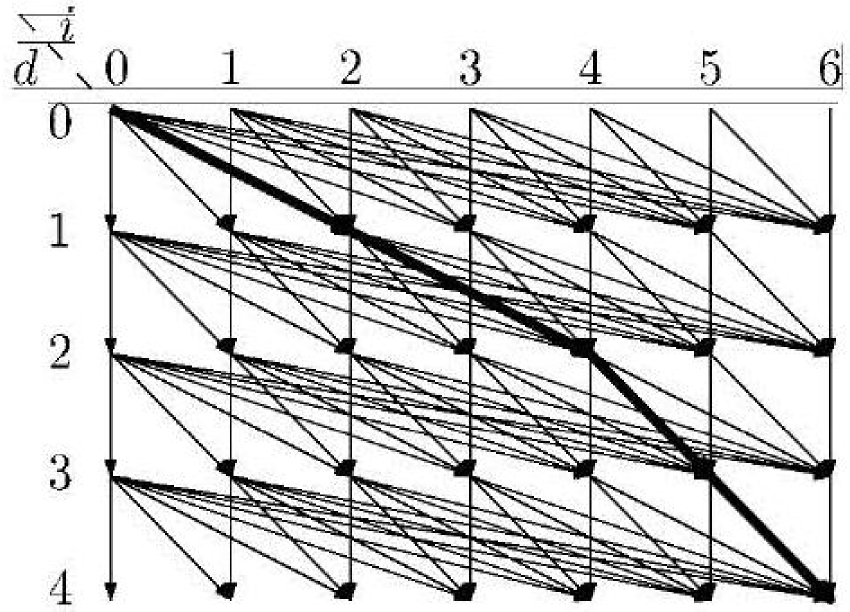

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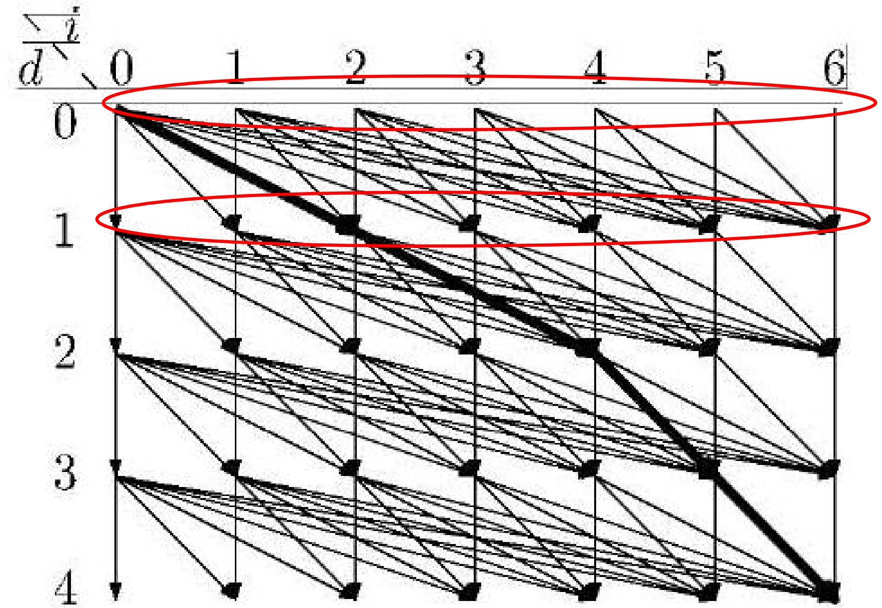

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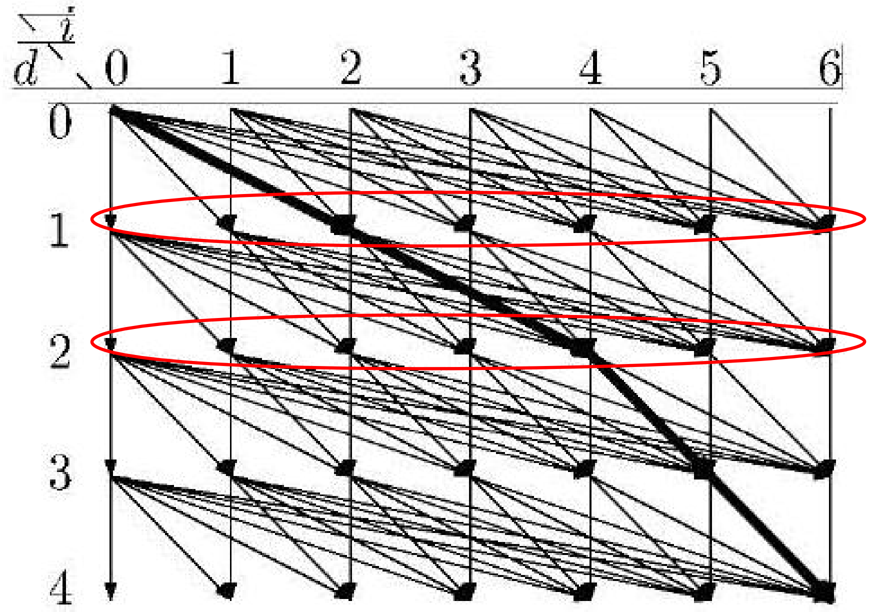

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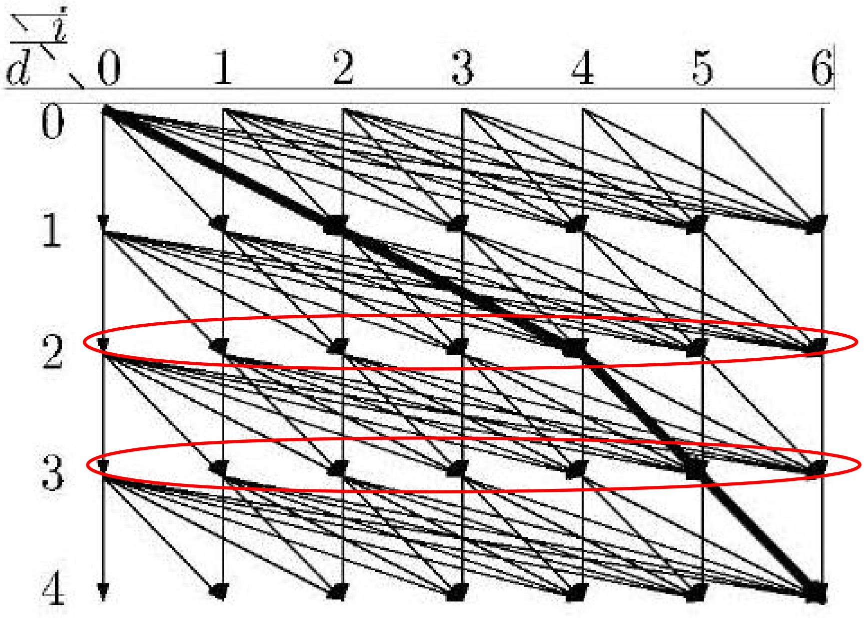

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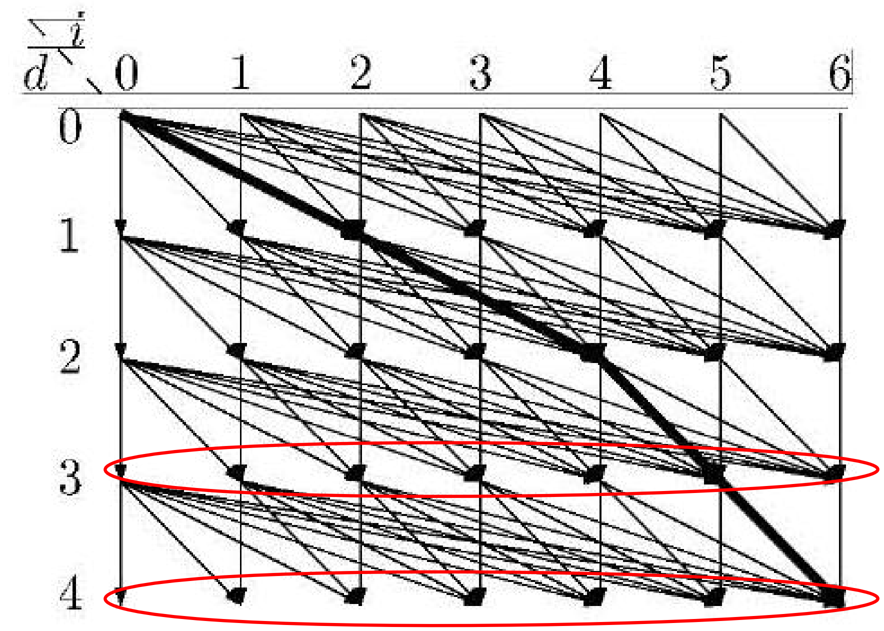

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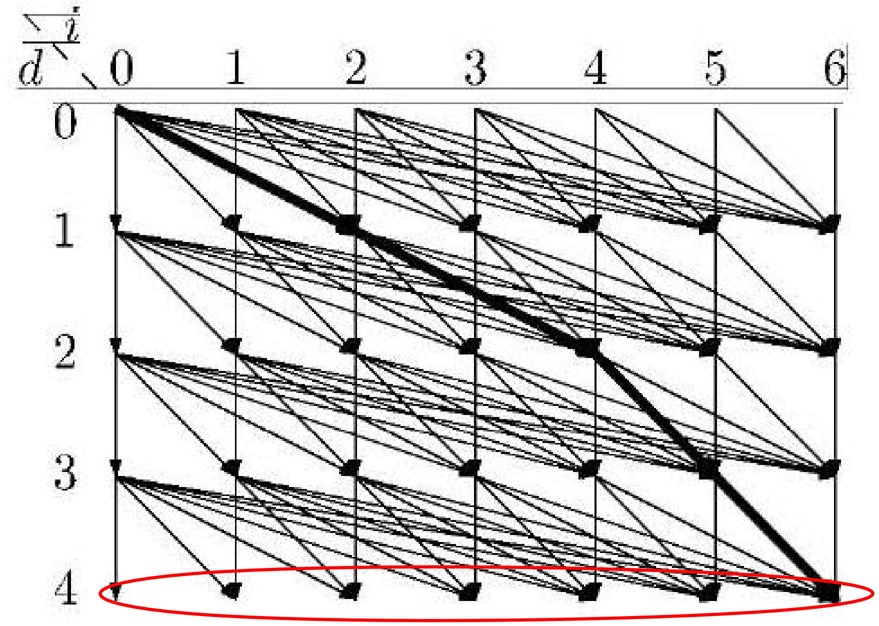

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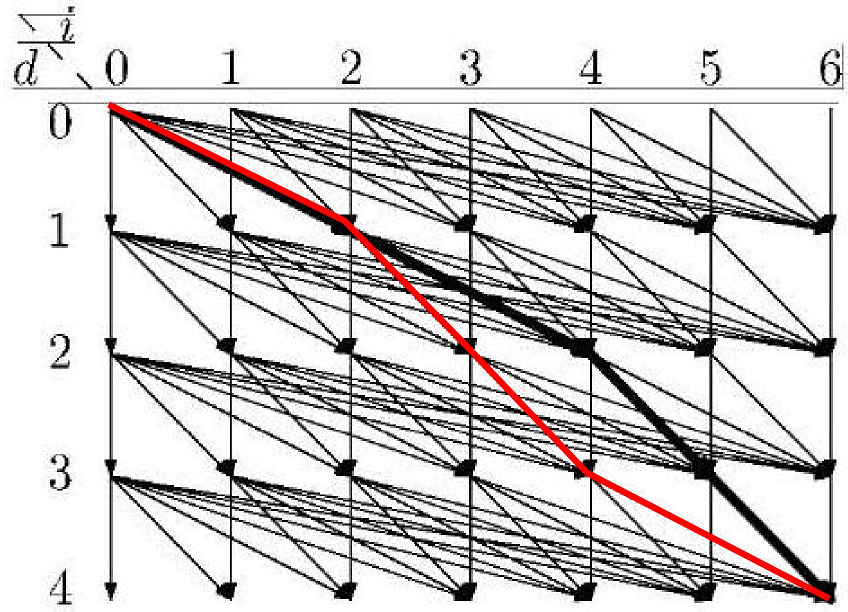

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On the other hand, finding optimal path to $H(D,n)$ requires keeping entire $\Theta(nD)$ space table to backtrack through

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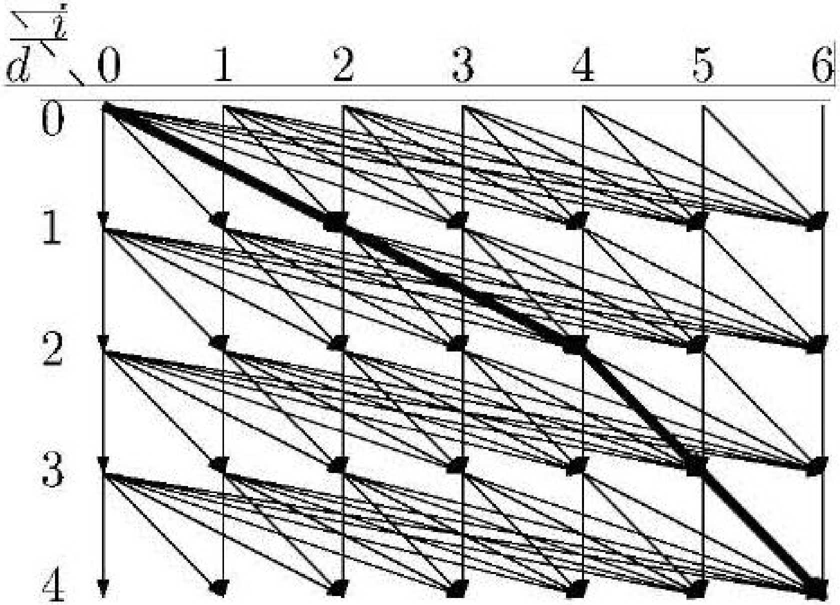
We will now see how to find path using $O(D+n)$ space.

Modification of idea due to

Hirschberg ('75)

Munro & Ramirez ('82)



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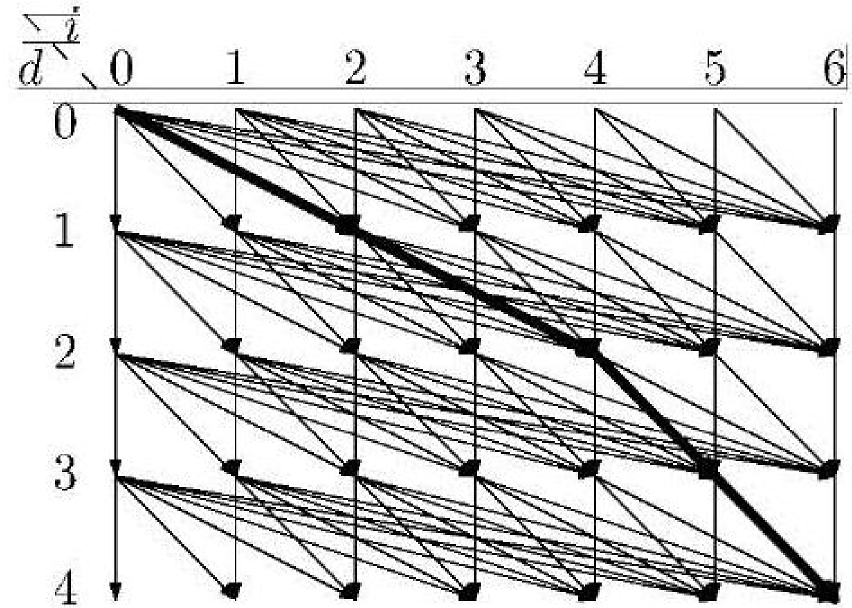

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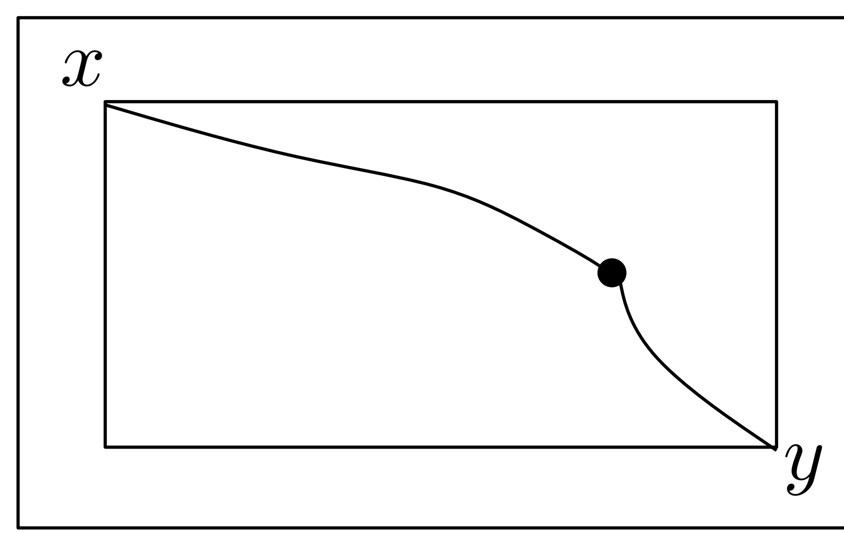
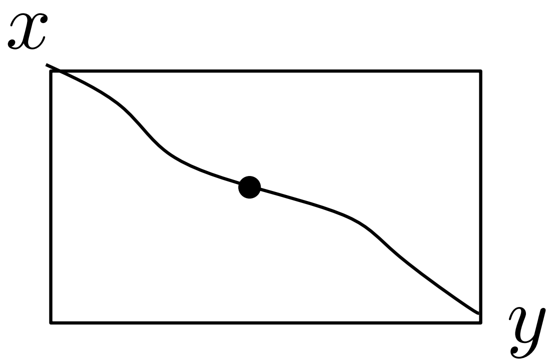
Let y be below and to the right of x. Assume existence of an oracle $Mid(x,y)$ that returns a midpoint (hop distance) on some min-cost x-y path.



$Mid(x, y)$ returns a midpoint (hop distance) on some min-cost x-y path.



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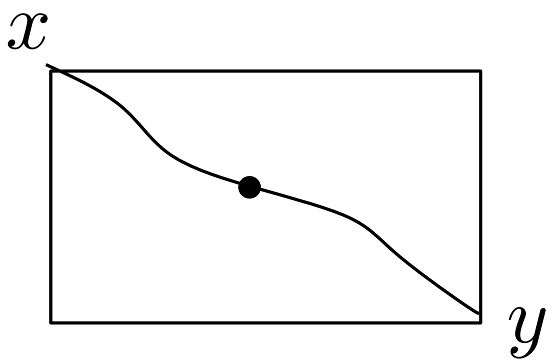

We now have a simple recursive procedure for building min-cost path

Buildpath(x,y)

If $y_d = x_{d+1}$

return $(x \to y)$

else

$z = Mid(x, y)$

Buildpath(x,z)

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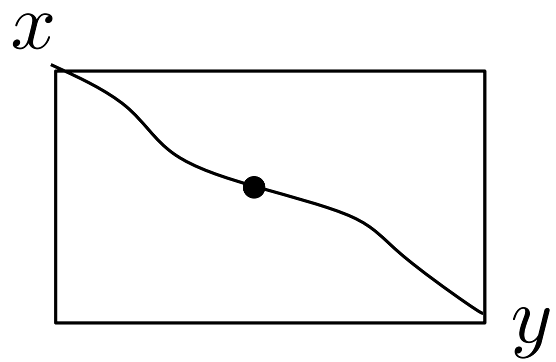

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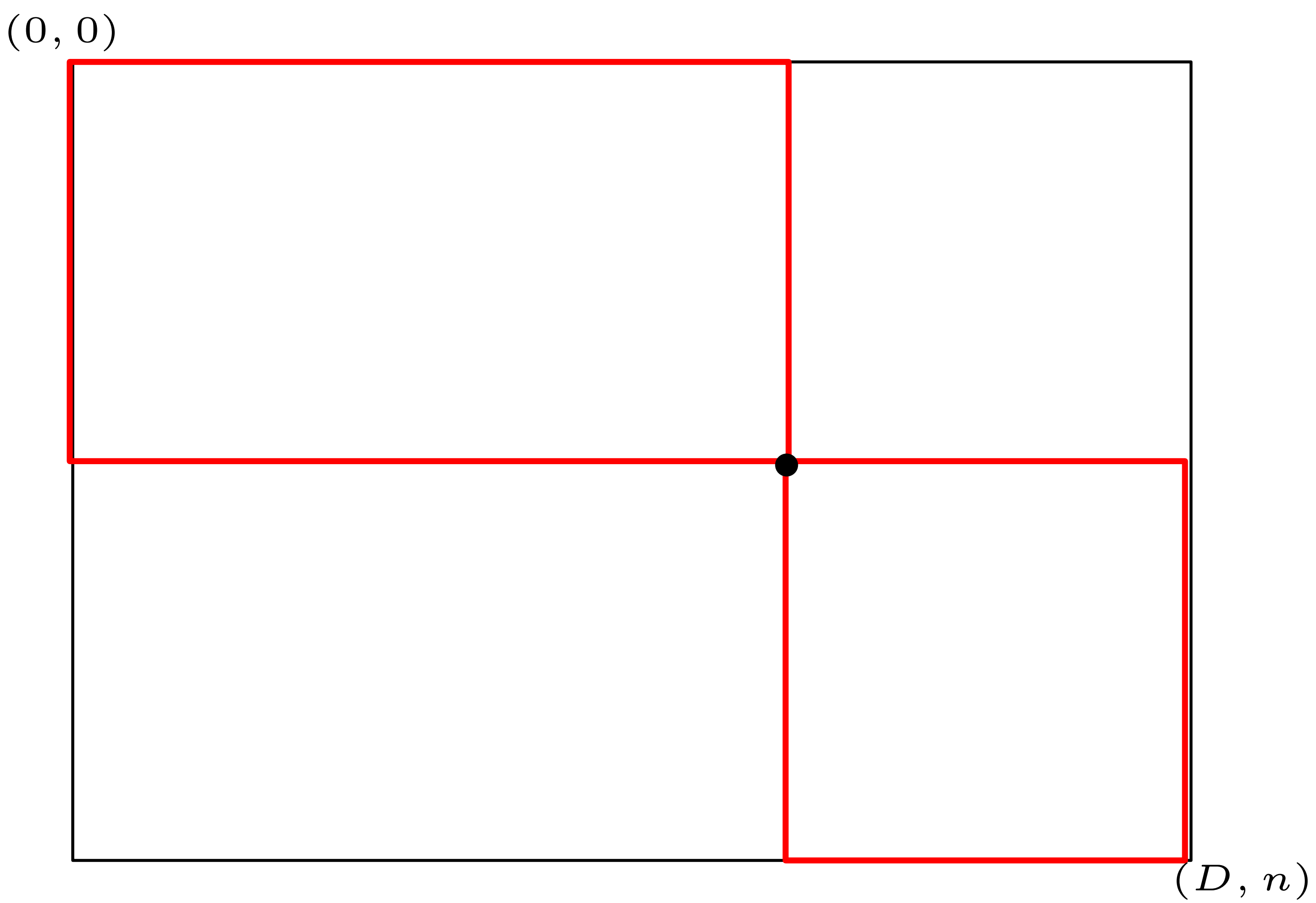

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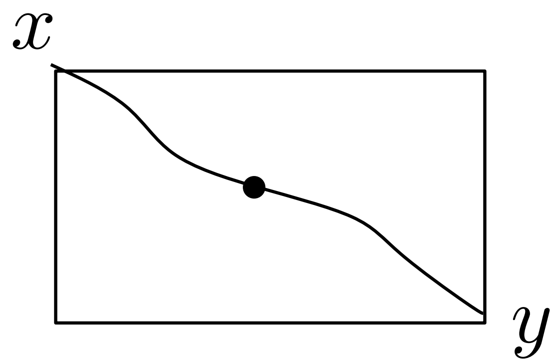

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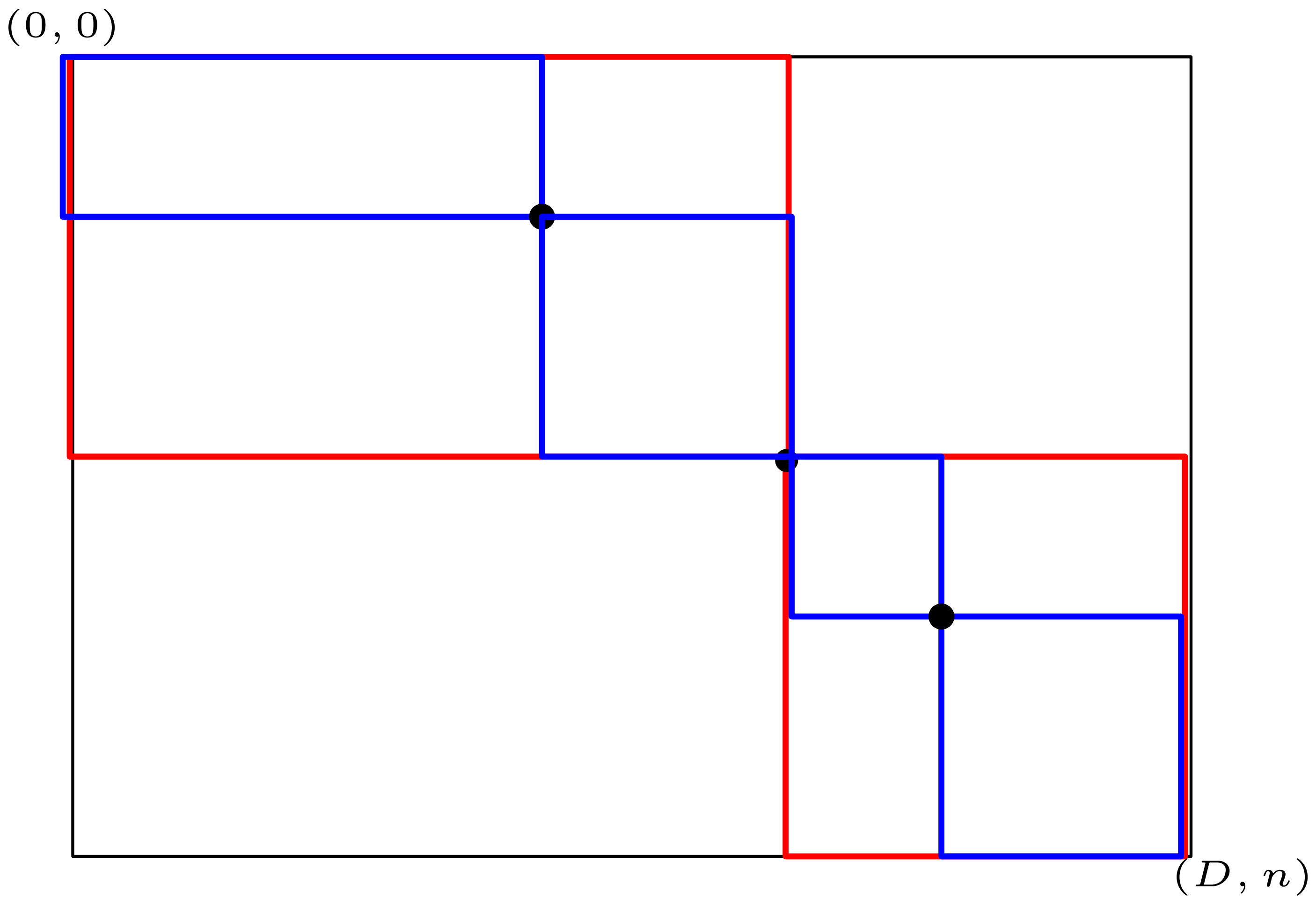

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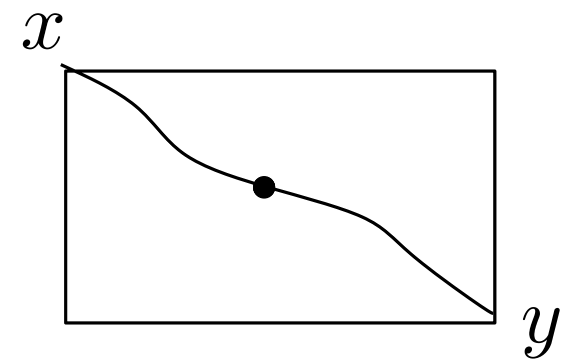

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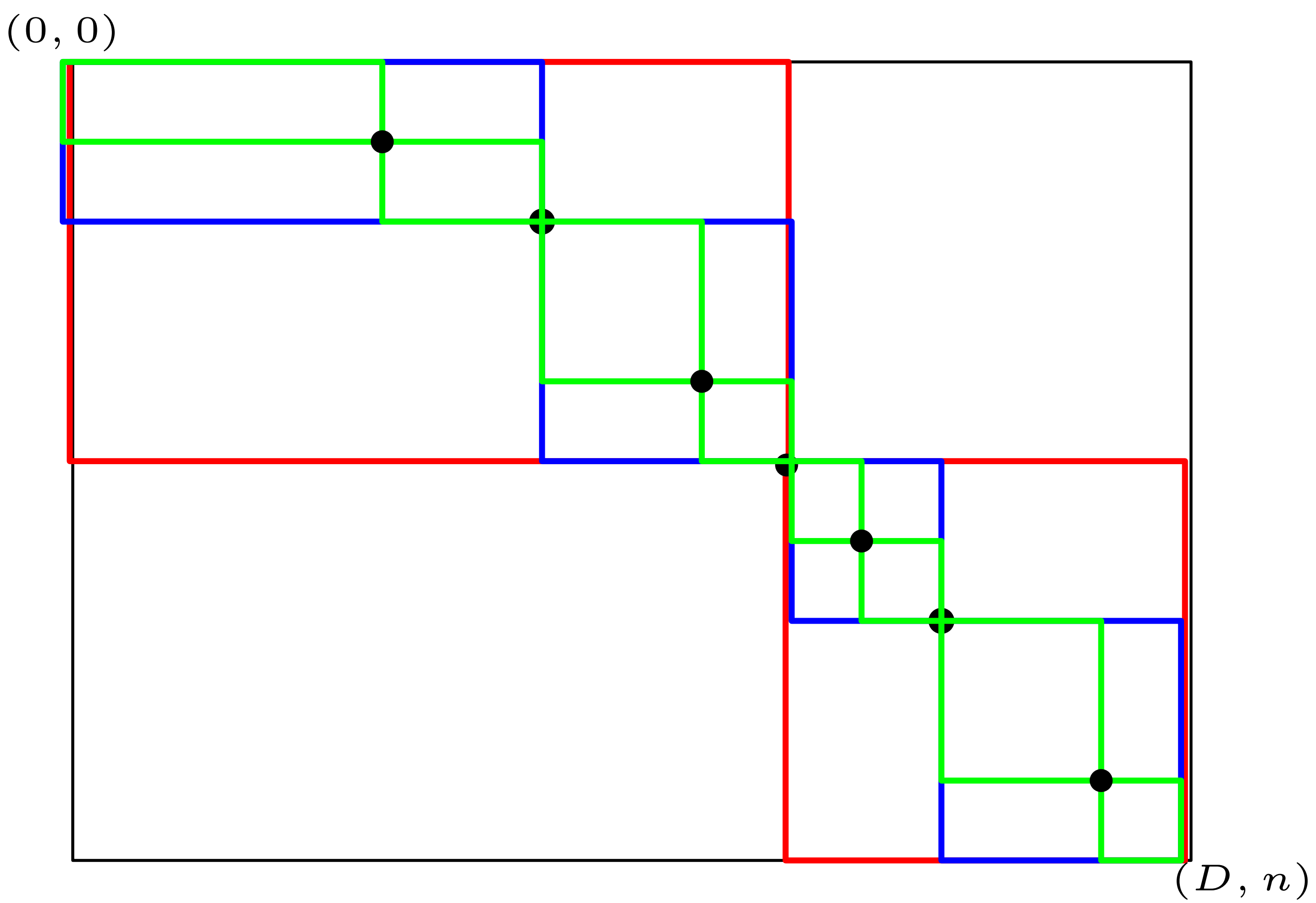

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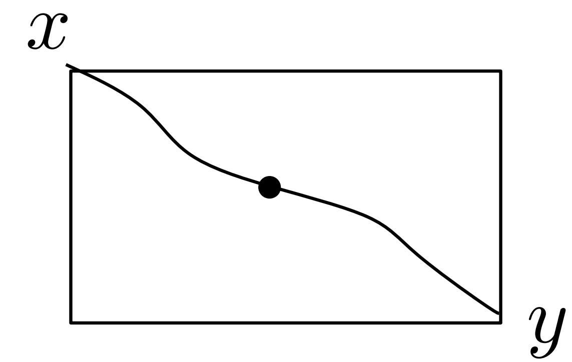

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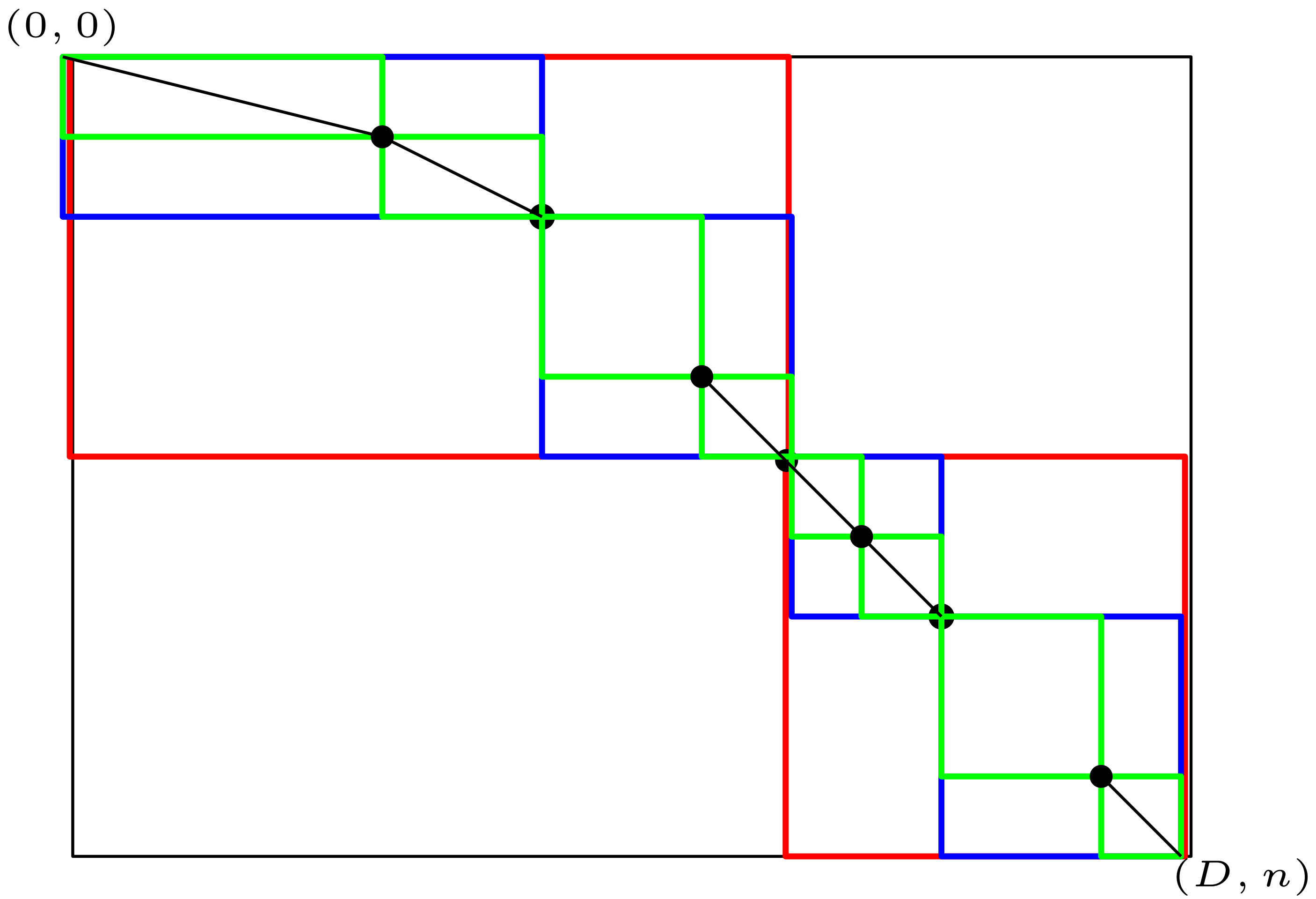

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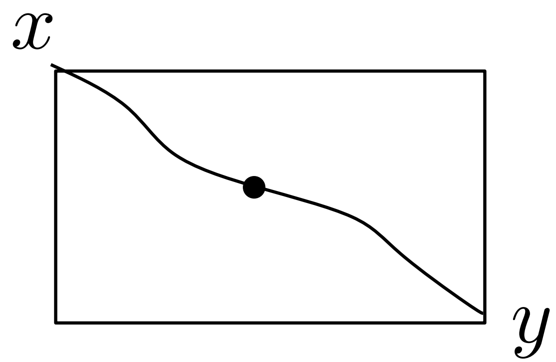

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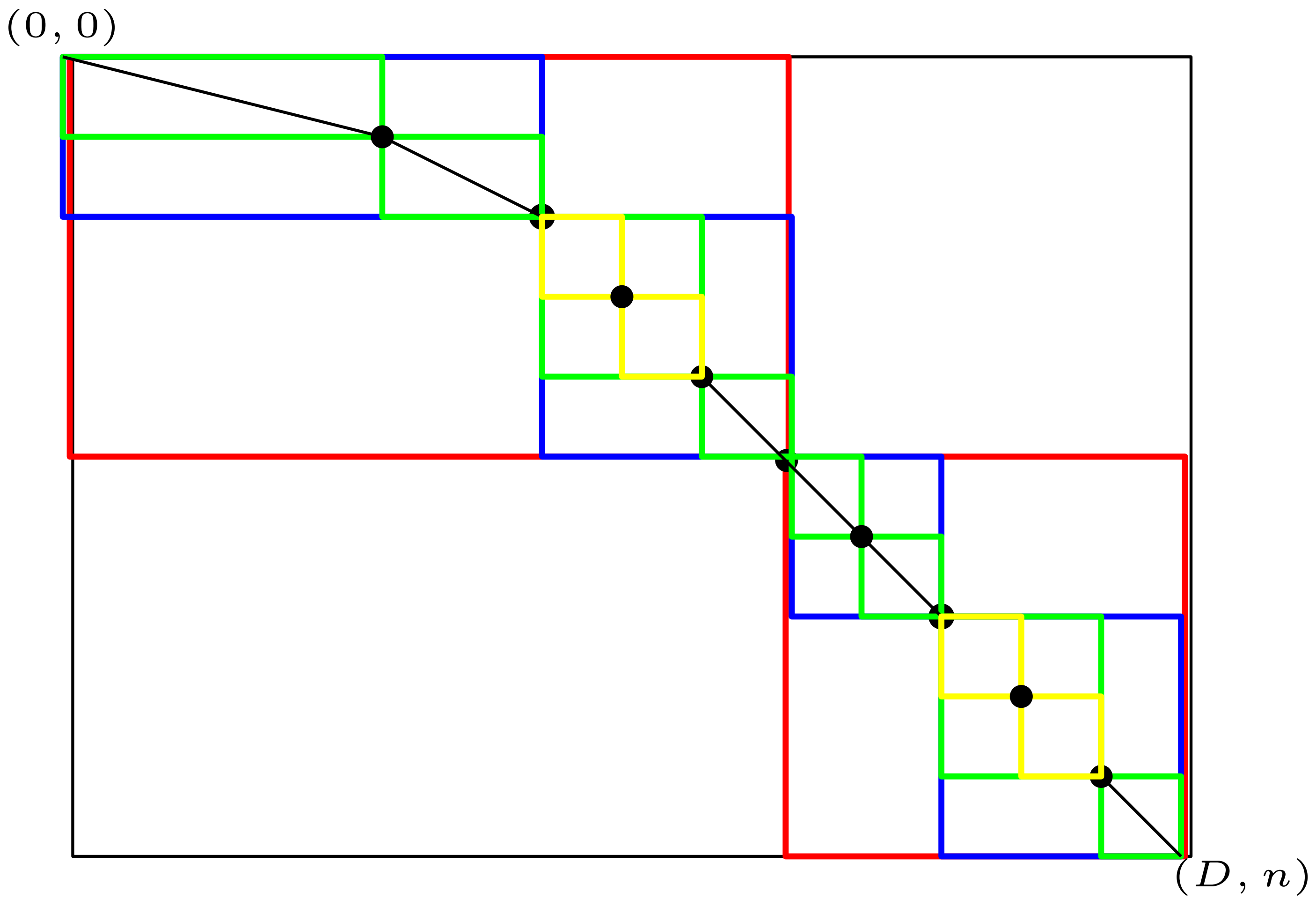

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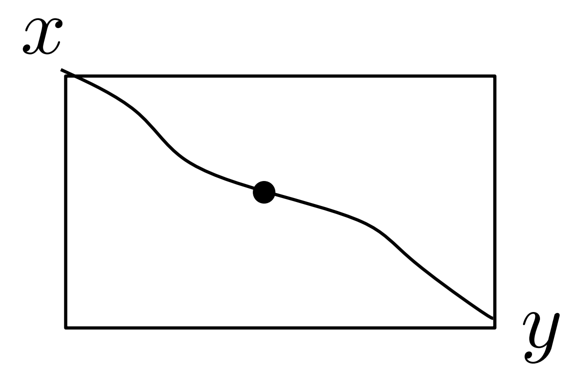

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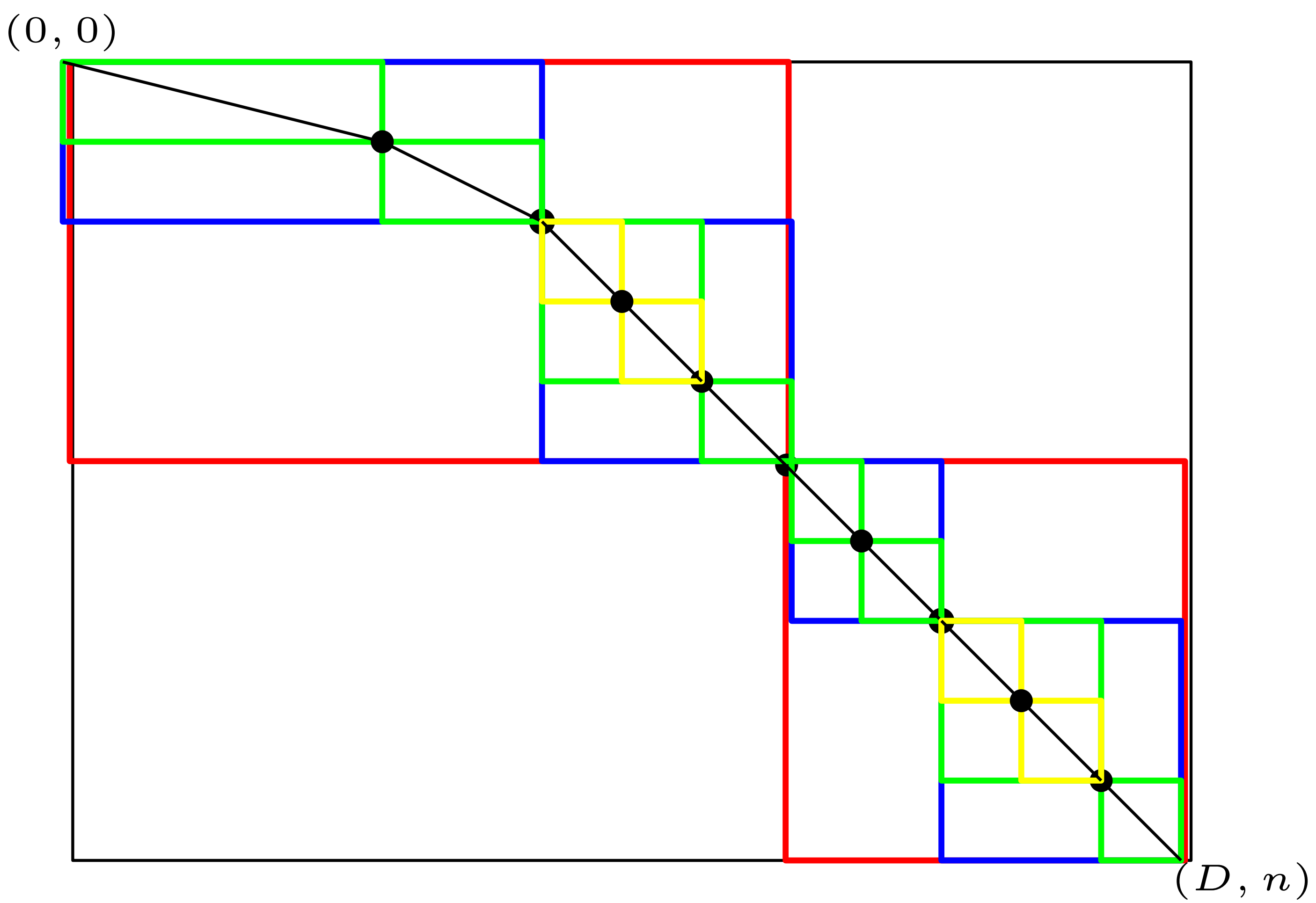

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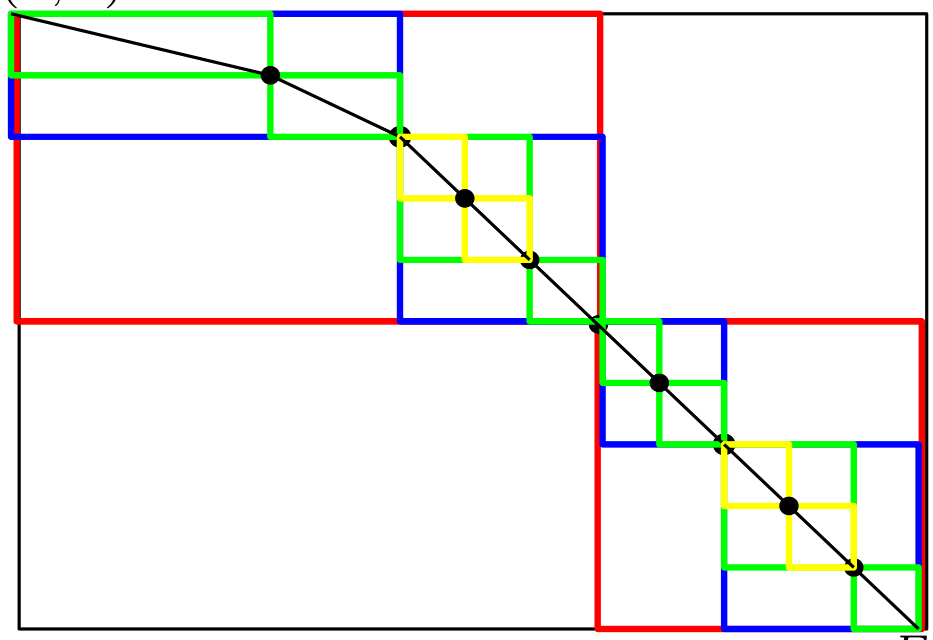

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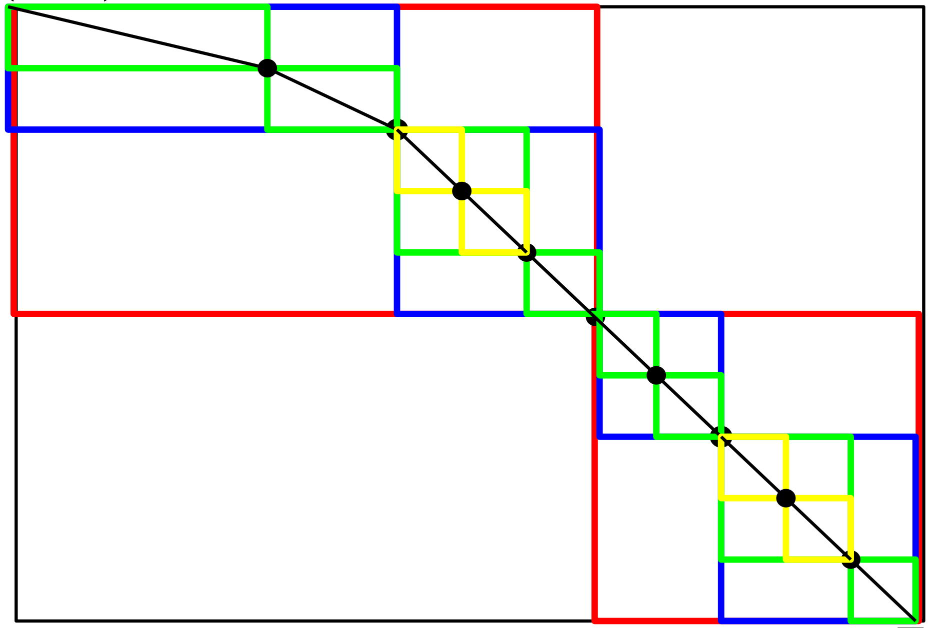

Lemma: If $Mid(x, y)$ uses $O(D + n)$ space

$\Longrightarrow$ Buildpath(0,F) uses $O(D + n)$ space

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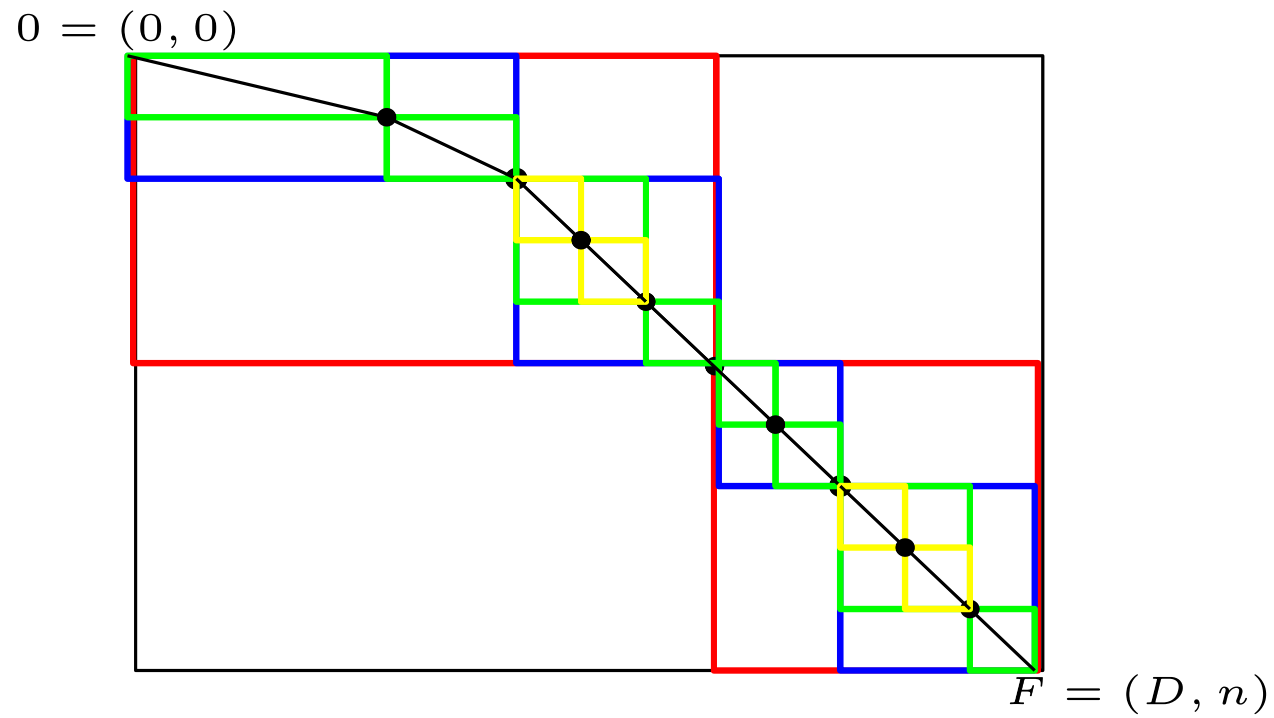

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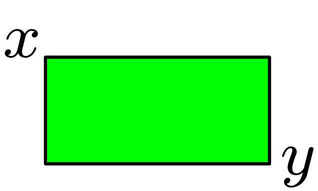
Lemma: Let $Area(x, y)$ be area of x, y box



If $Mid(x, y)$ uses $O(Area(x, y))$ time

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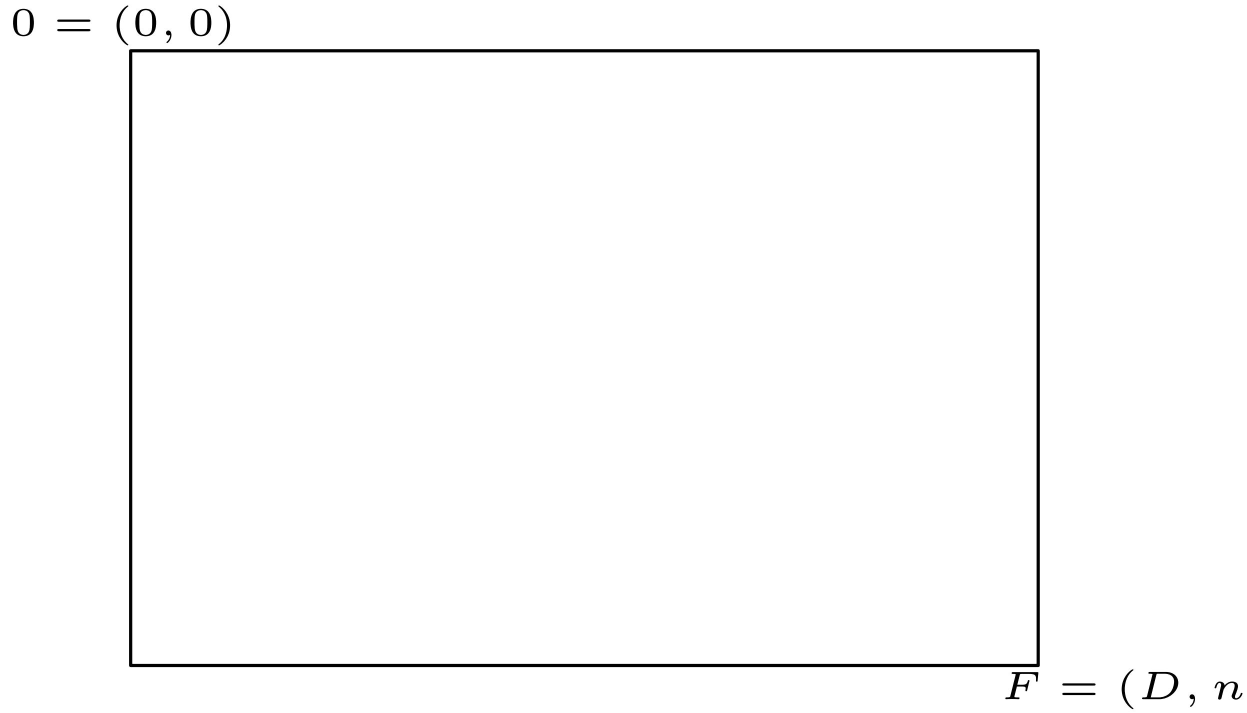

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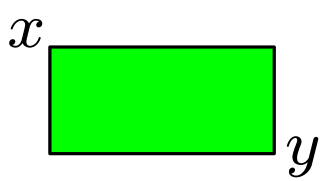

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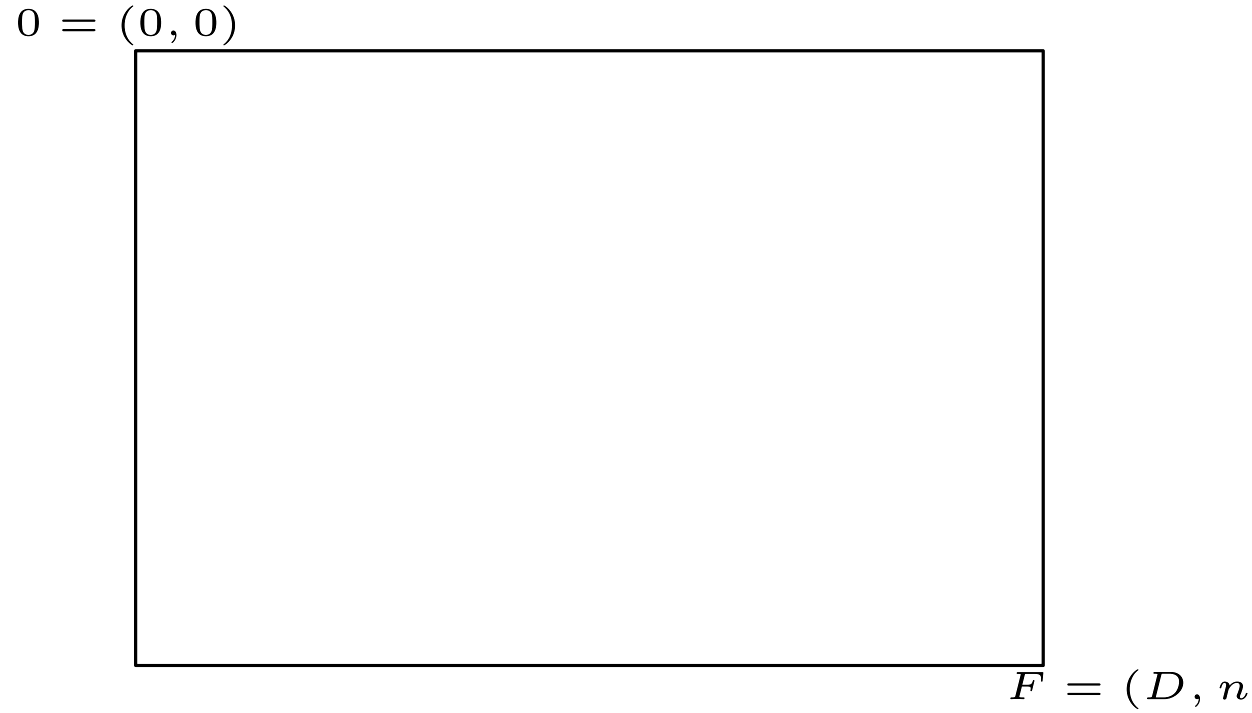

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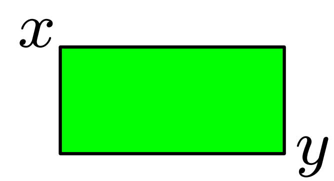

Lemma: Let $Area(x, y)$ be area of x, y box

If $Mid(x, y)$ uses $O(Area(x, y))$ time

$\Longrightarrow$ Buildpath(0,F) uses $O(Dn)$ time

Proof: Rectangles at recursion level i are height $\leq D/2^i$

$\Longrightarrow$ Total work at level i is $\leq nD/2^i$

$\Longrightarrow$ Total work $\leq$

Buildpath(x,y)

If $y_d = x_{d+1}$
  return $(x \longrightarrow y)$
else
  $z = Mid(x, y)$
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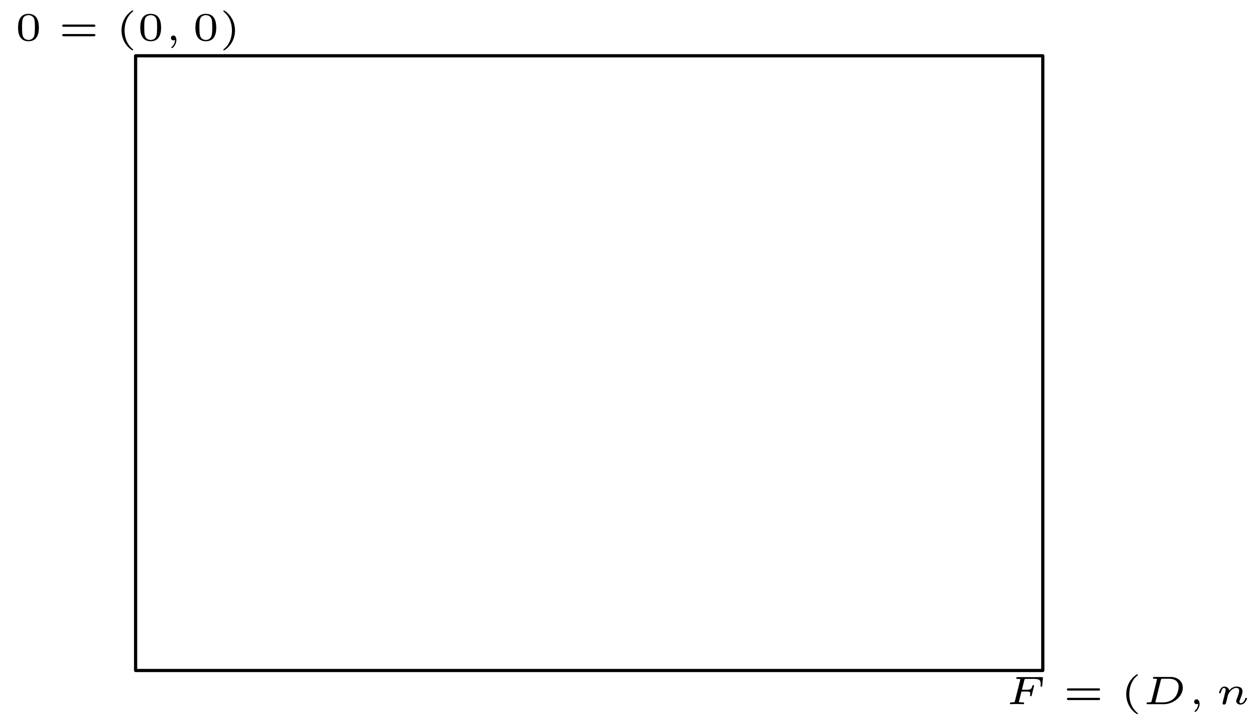

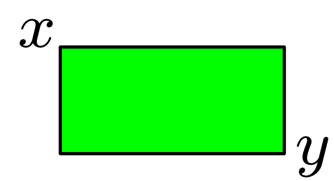

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If $Mid(x, y)$ uses $O(Area(x, y))$ time

$\Longrightarrow$ Buildpath(0,F) uses $O(Dn)$ time

Proof: Rectangles at recursion level i are height $\leq D/2^i$

$\Longrightarrow$ Total work at level i is $\leq nD/2^i$

$\Longrightarrow$ Total work $\leq n\left(\frac{D}{2^0}\right.$

Buildpath(x,y)

If $y_d = x_{d+1}$
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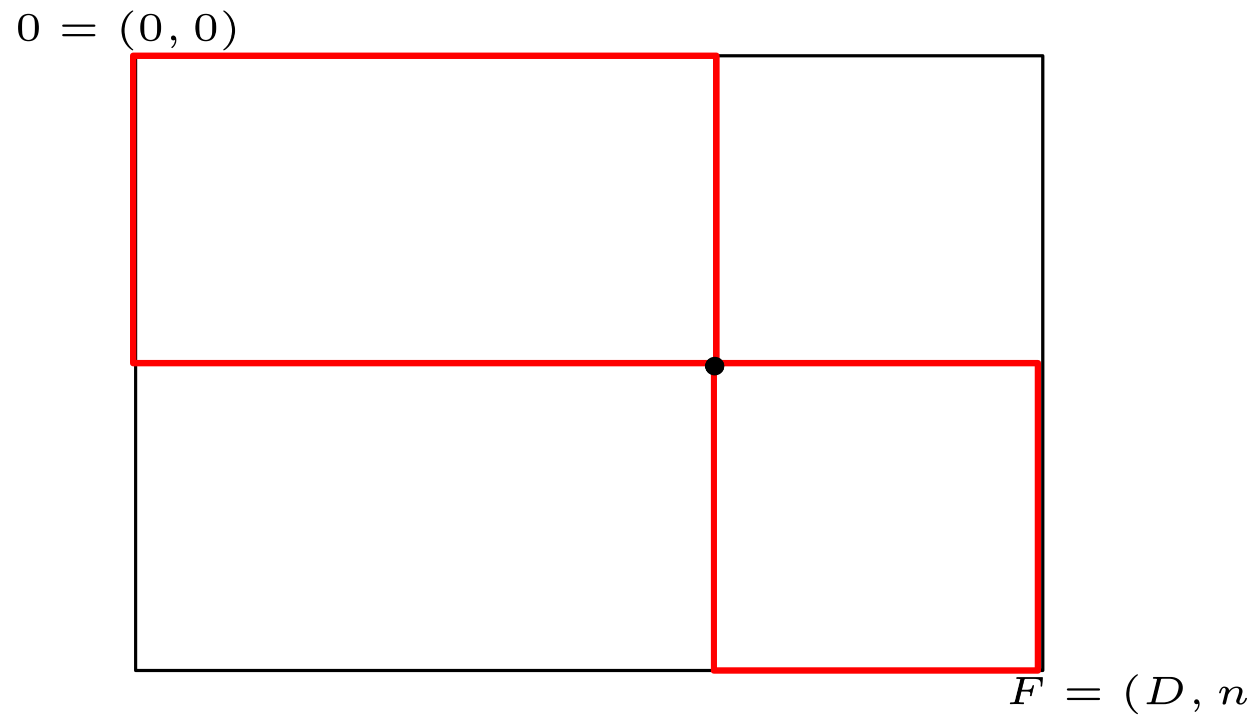

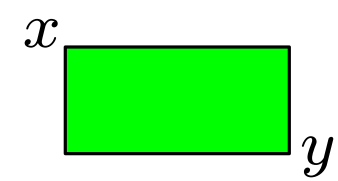

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Proof: Rectangles at recursion level i are height $\leq D/2^i$

$\Longrightarrow$ Total work at level i is $\leq nD/2^i$

$\Longrightarrow$ Total work $\leq n \left(\frac{D}{2^0} + \frac{D}{2^1} \right.$

Buildpath(x,y)

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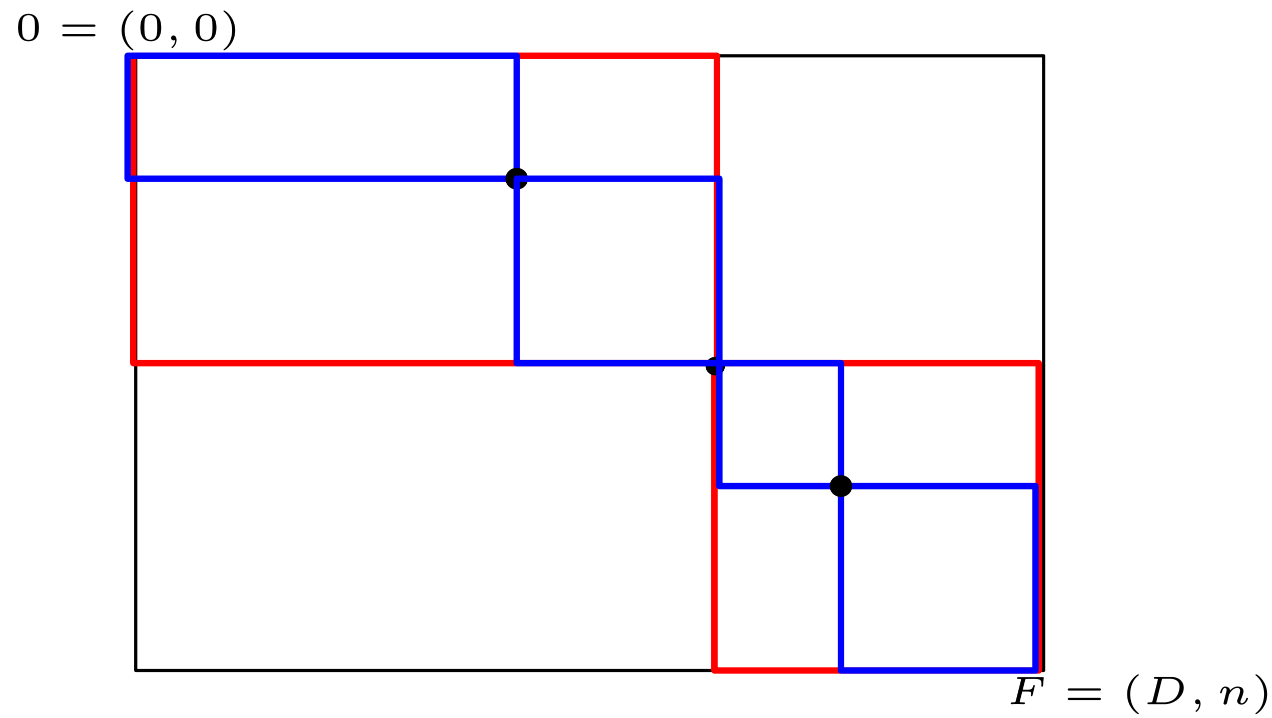

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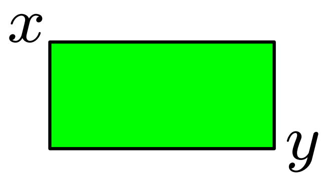

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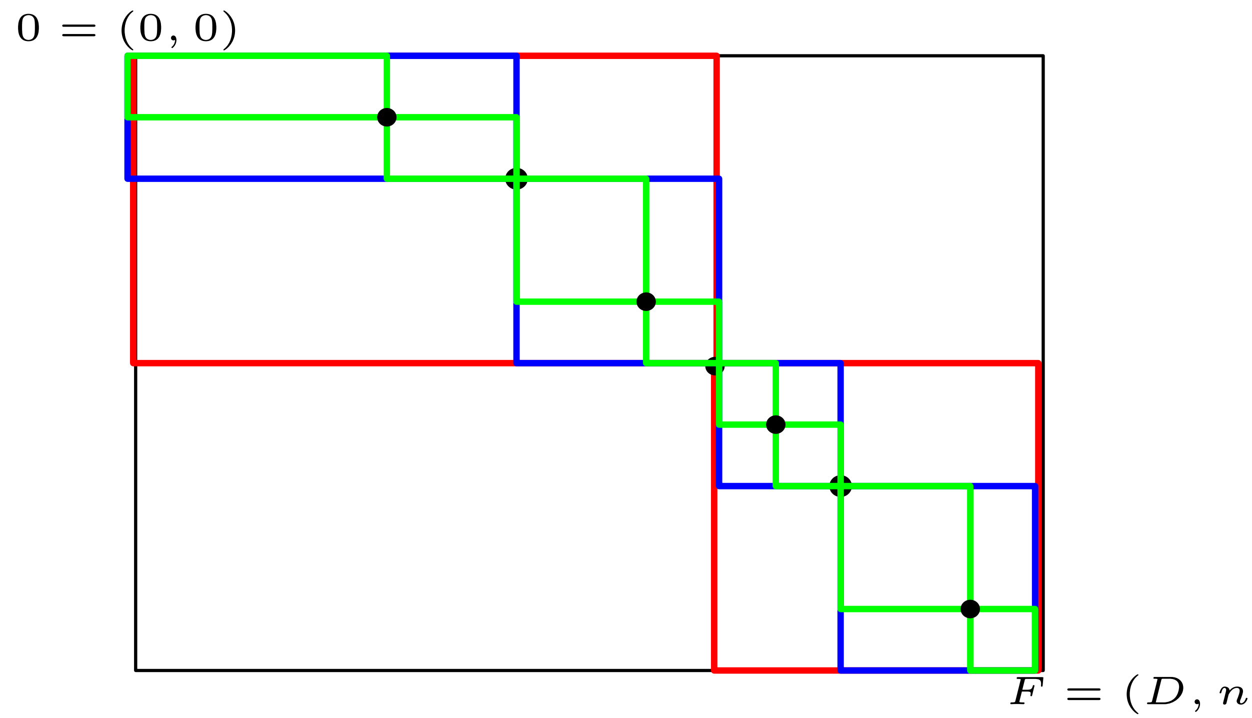
$\Longrightarrow$ Buildpath(0,F) uses $O(Dn)$ time

Proof: Rectangles at recursion level i are height $\leq D/2^i$

$\Longrightarrow$ Total work at level i is $\leq nD/2^i$

$\Longrightarrow$ Total work $\leq n\left(\frac{D}{2^0} + \frac{D}{2^1} + \frac{D}{2^2}\right.$



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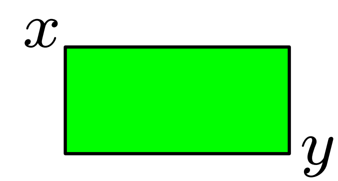

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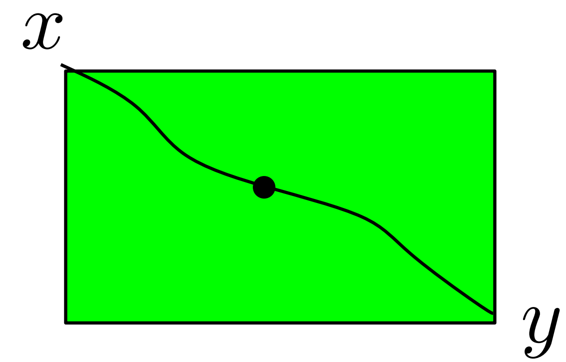
$\Longrightarrow$ Total work at level i is $\leq nD/2^i$

$\Longrightarrow$ Total work $\leq n\left(\frac{D}{2^0} + \frac{D}{2^1} + \frac{D}{2^2} + \frac{D}{2^3} + \cdots\right) \leq 2nD$

Just saw that if $Mid(x,y)$ can be implemented using $O(D+n)$ space and $Area(x,y)$ time, then path can be built using $O(D+n)$ space and $O(Dn)$ time.



There are two different methods in literature for implementing $Mid(x,y)$. They can both be used here, but we will use (b).

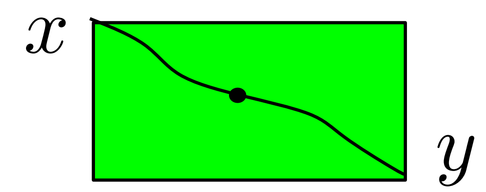
(a) Hirschberg ('75)

For longest common subsequence problem.
Runs two modified Dijkstra's that meet in "middle"
Every vertex had constant outdegree (≤ 3)
Used extensively in bioinformatics.

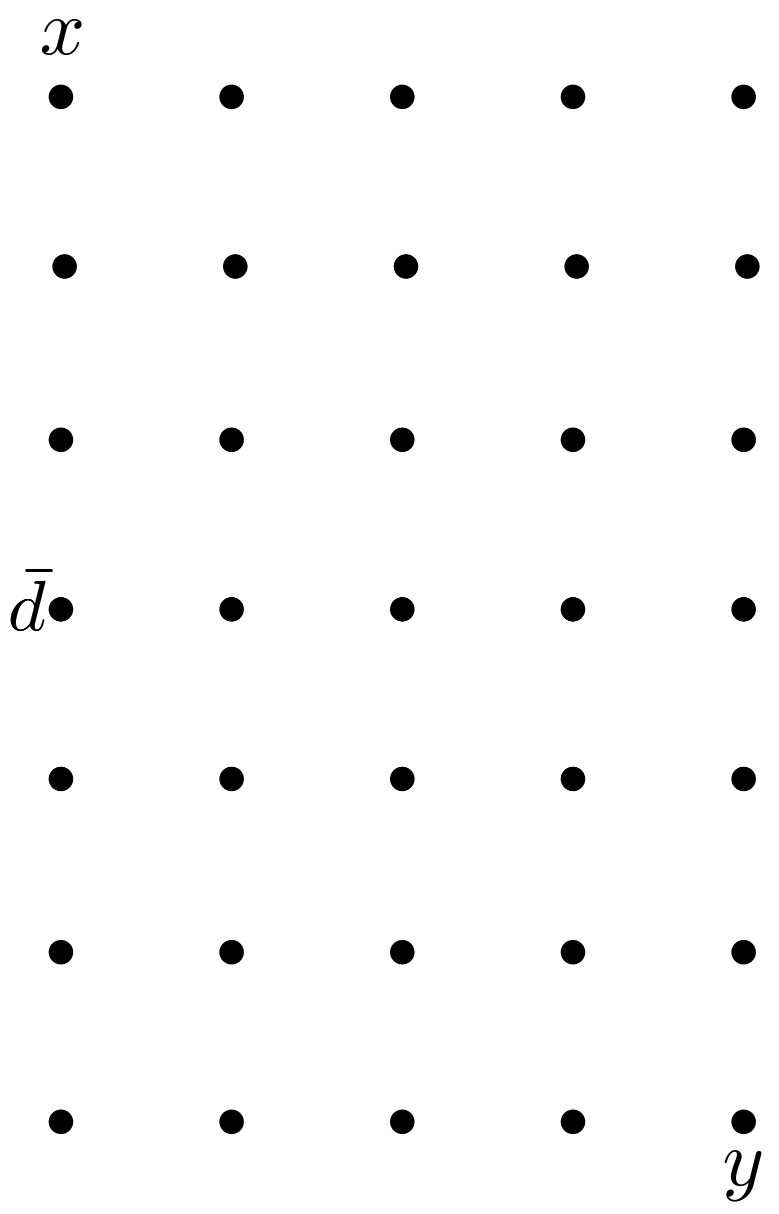
(b) Munro & Ramirez ('82)

For graphs like our's
Runs one modified Dijkstra
Uses $\Theta(Dn^2)$ time (we can improve to $\Theta(Dn)$ with Monge)

Implementing $Mid(x, y)$ in $O(D + n)$ space and $Area(x, y)$ time



For every z, let $C(z)$ be min cost path distance from x to z.
For $z_d \geq \bar{d}$, let $P(z)$ be a point on level $\bar{d}$ lying on some min-cost path.



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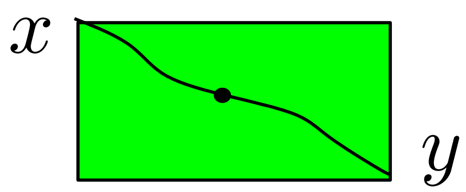

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If $z_d = \bar{d}$, $P(z) = z$.
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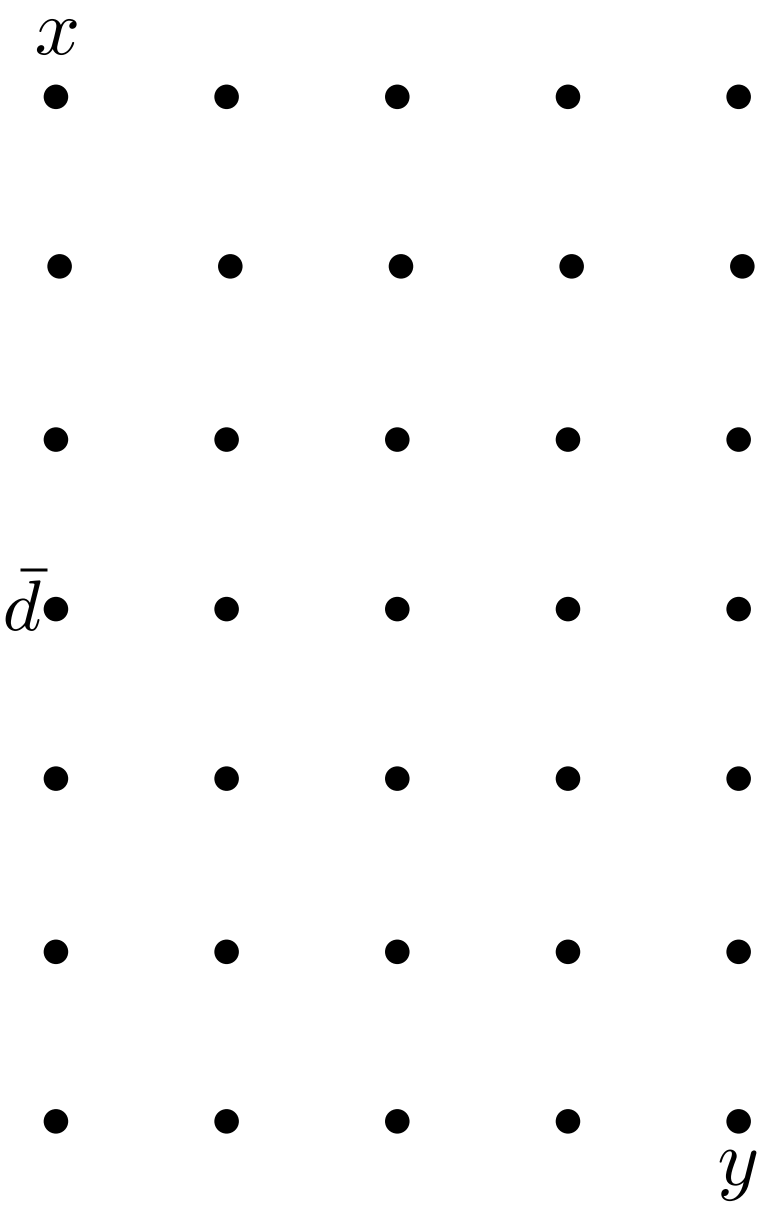

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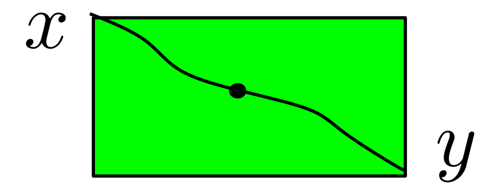

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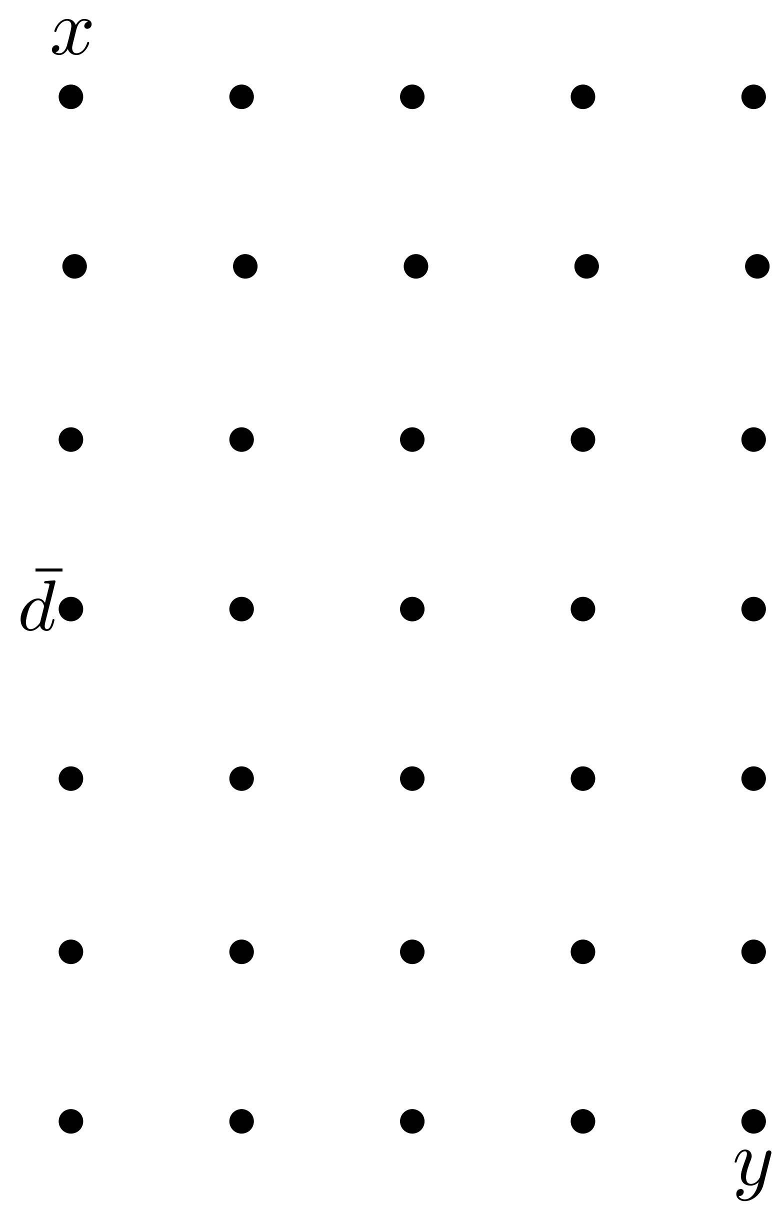

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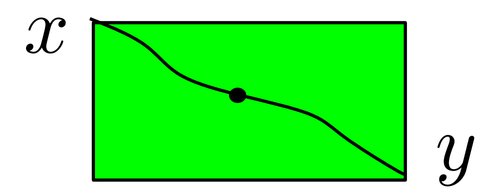
All of the $C(z)$ and $P(z)$ on level d can be calculated in $O(y_d - x_d)$ time (Monge property) using only knowledge of $C(z')$ and $P(z')$ where z' on level $d-1$.

Implementing $Mid(x,y)$ in $O(D+n)$ space and $Area(x,y)$ time



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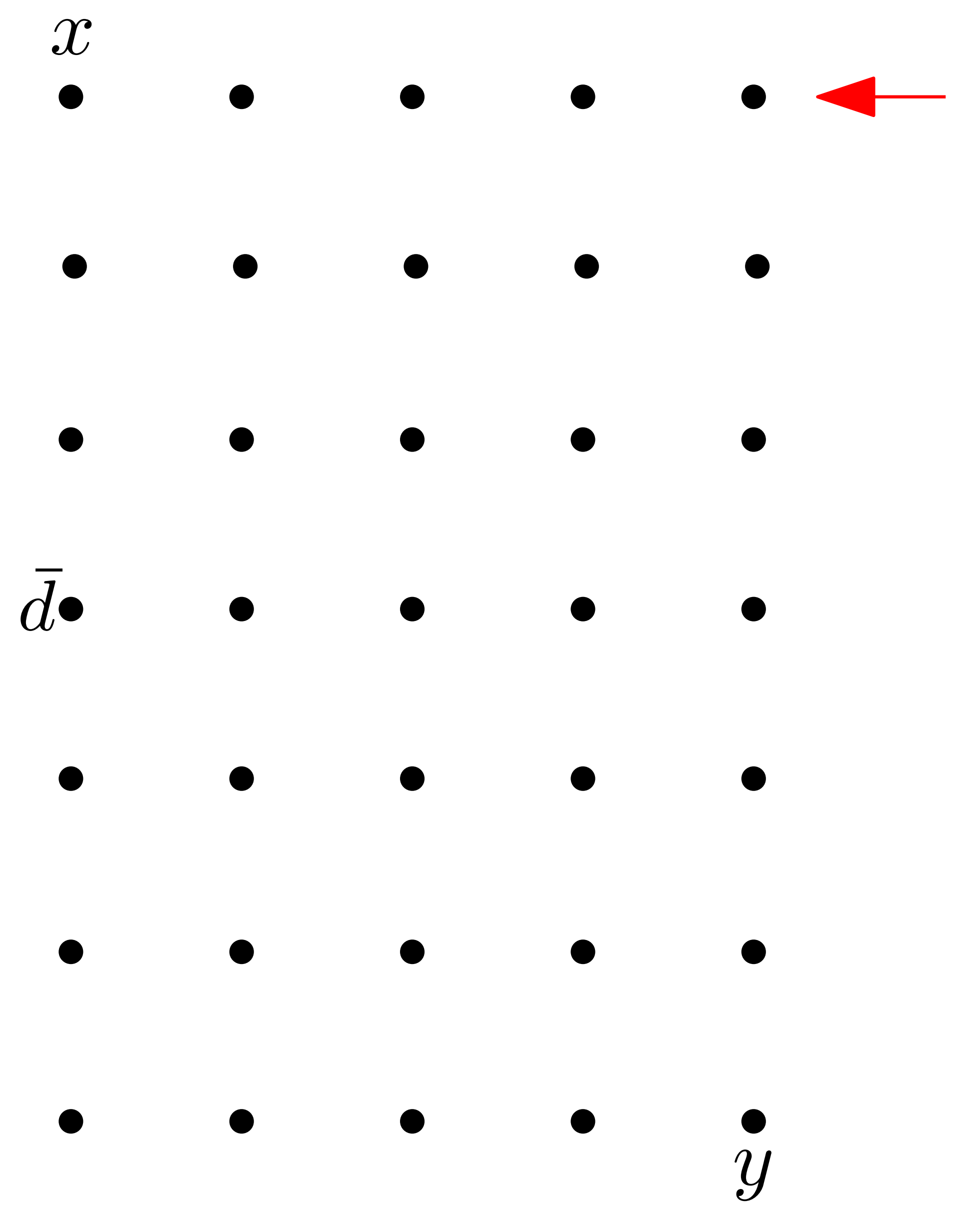

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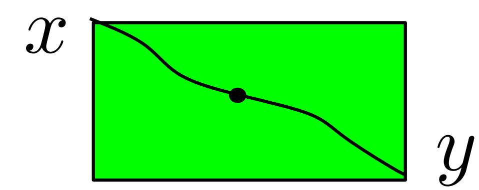

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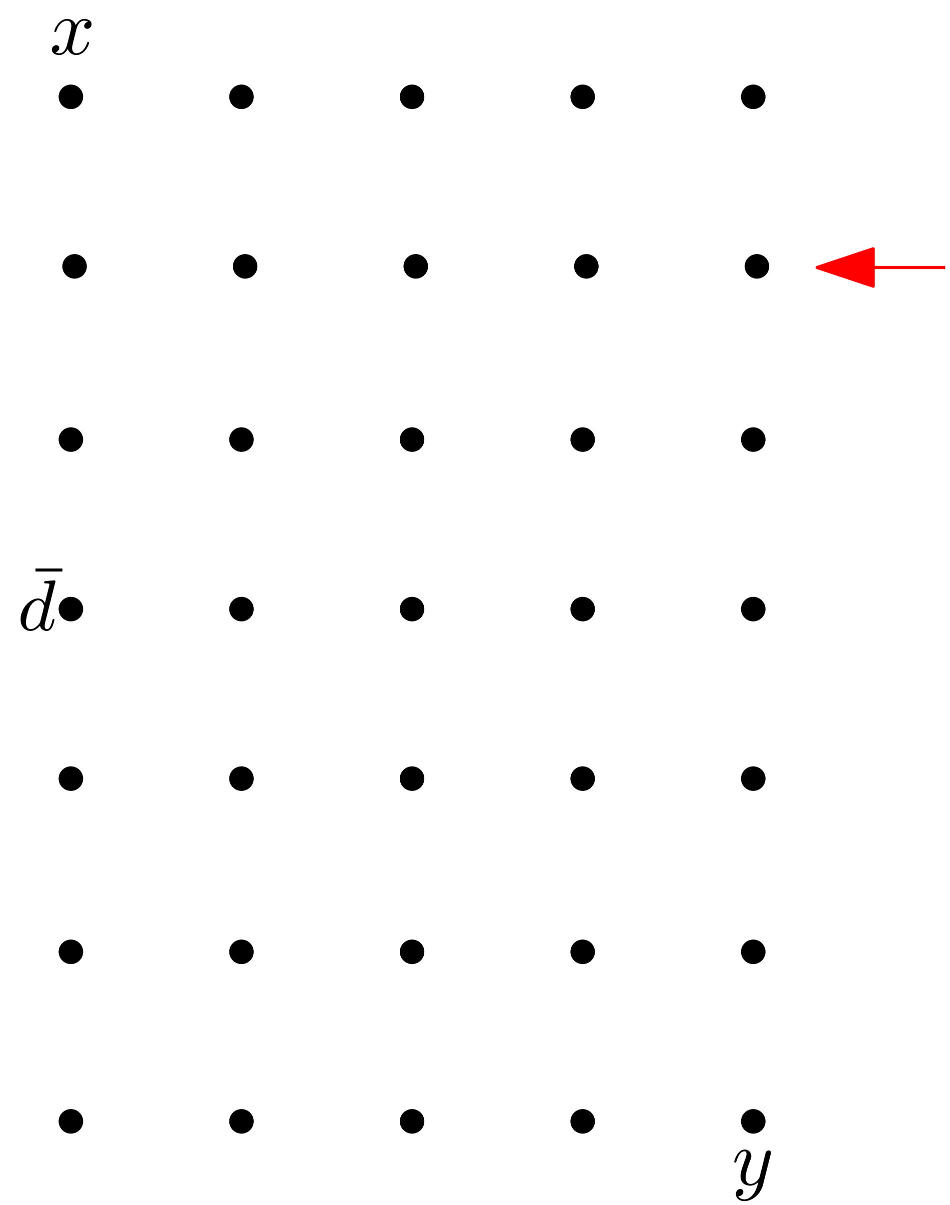

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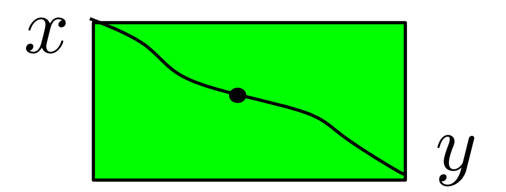

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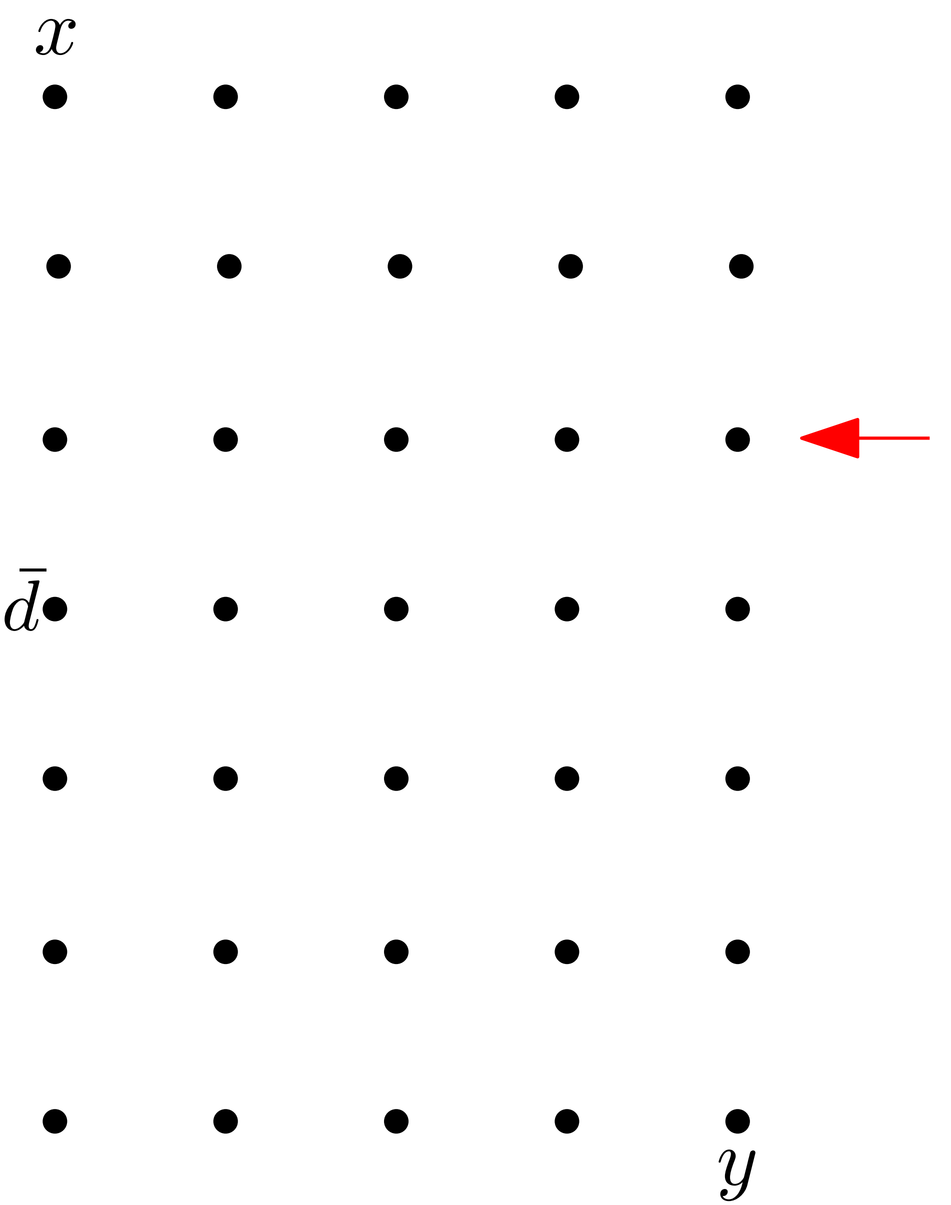

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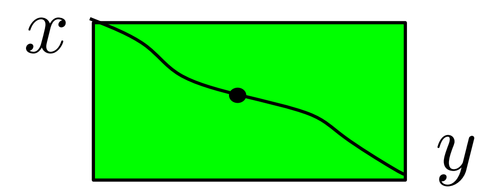

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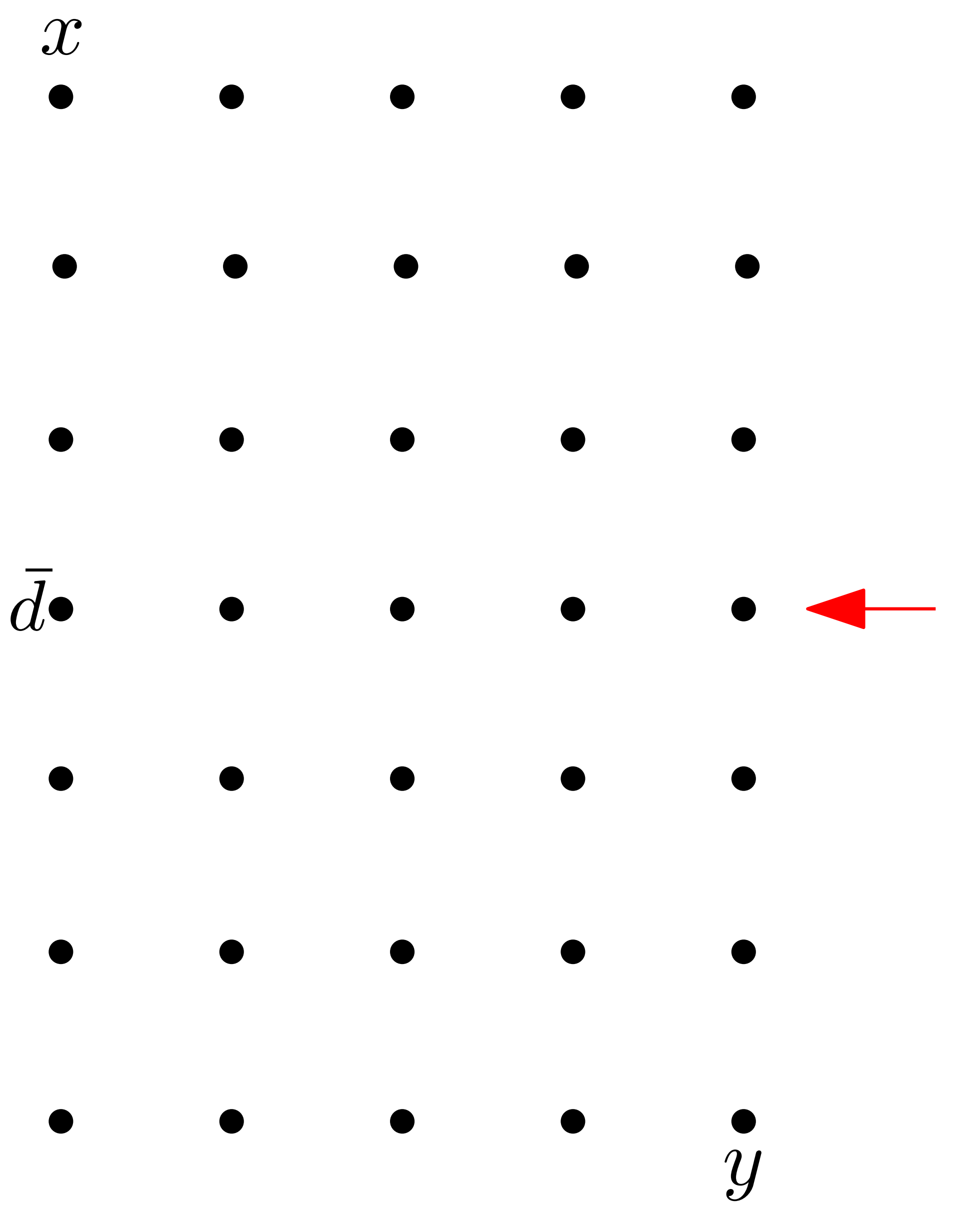

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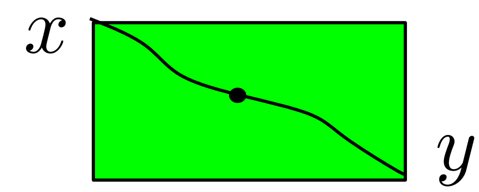

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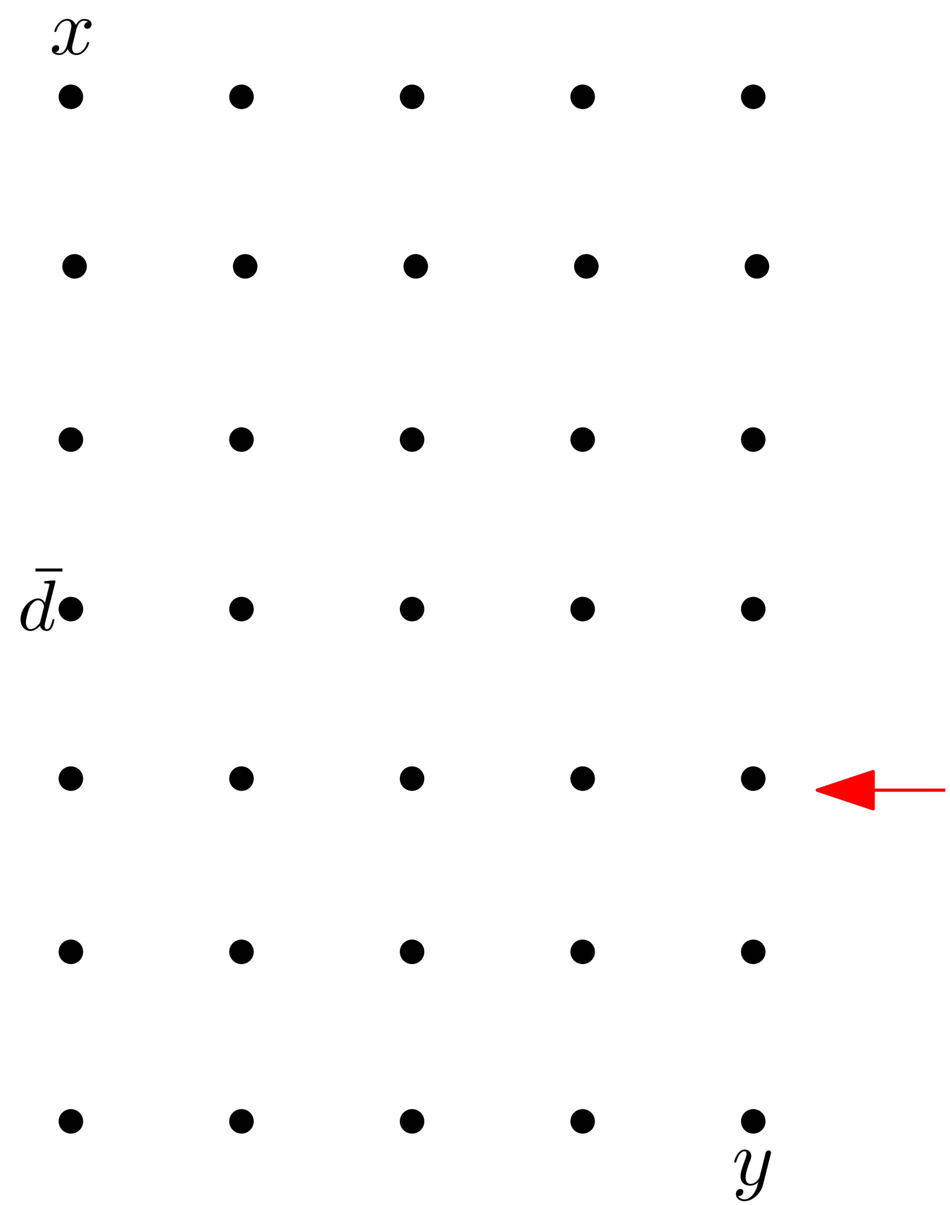

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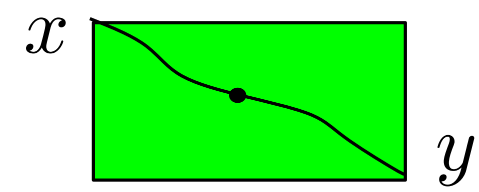

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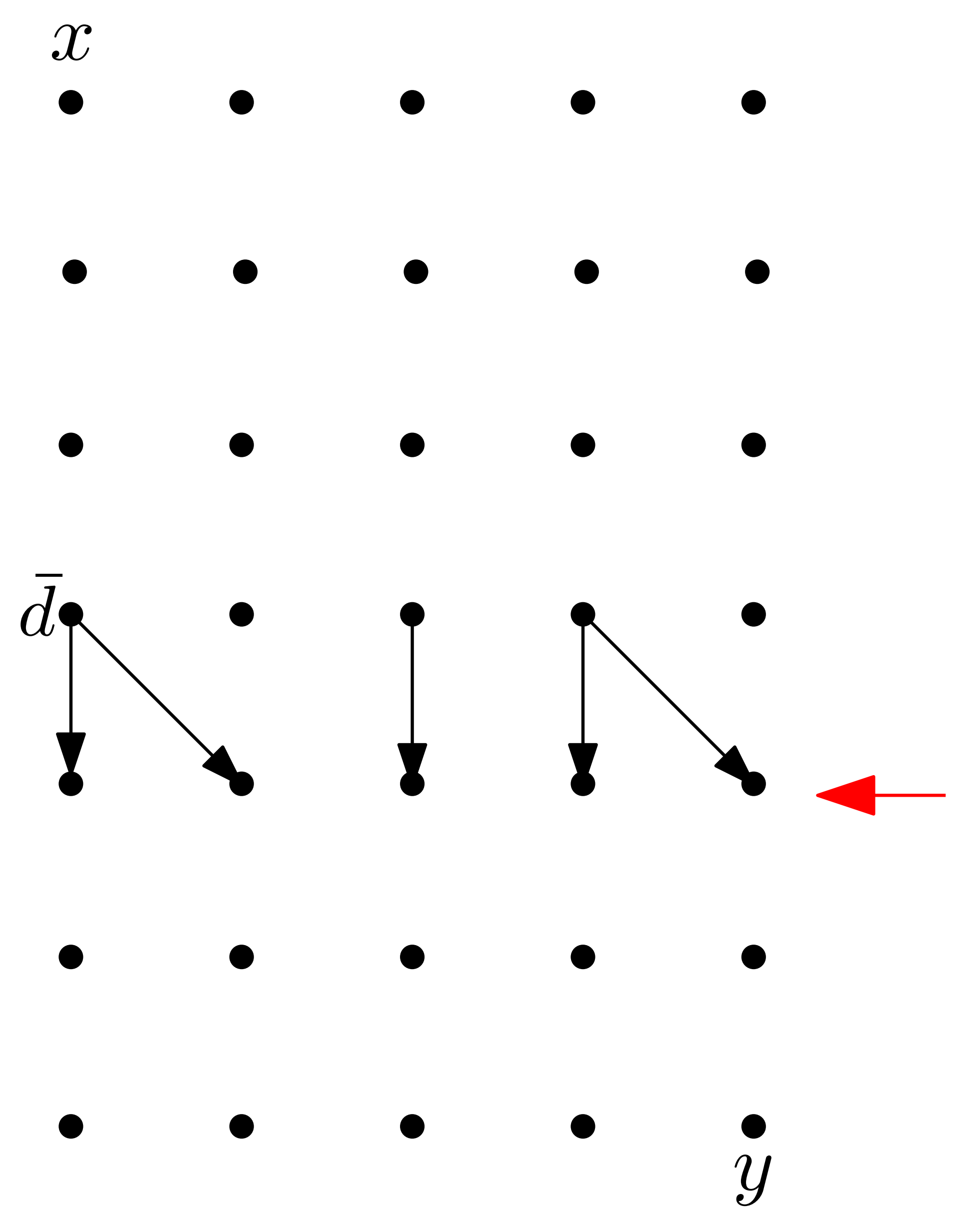

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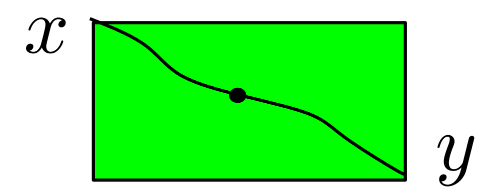

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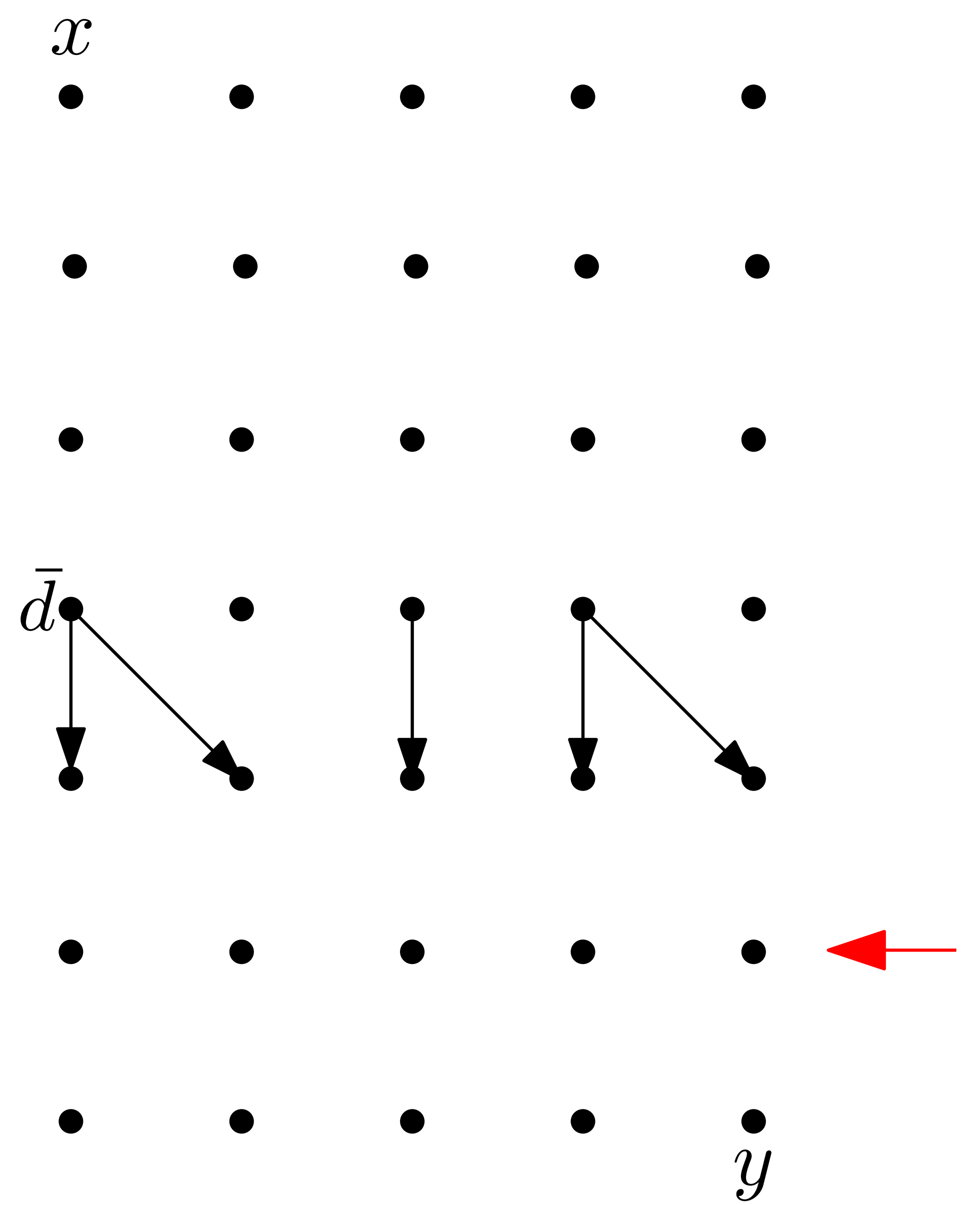

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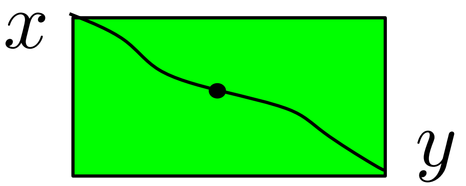

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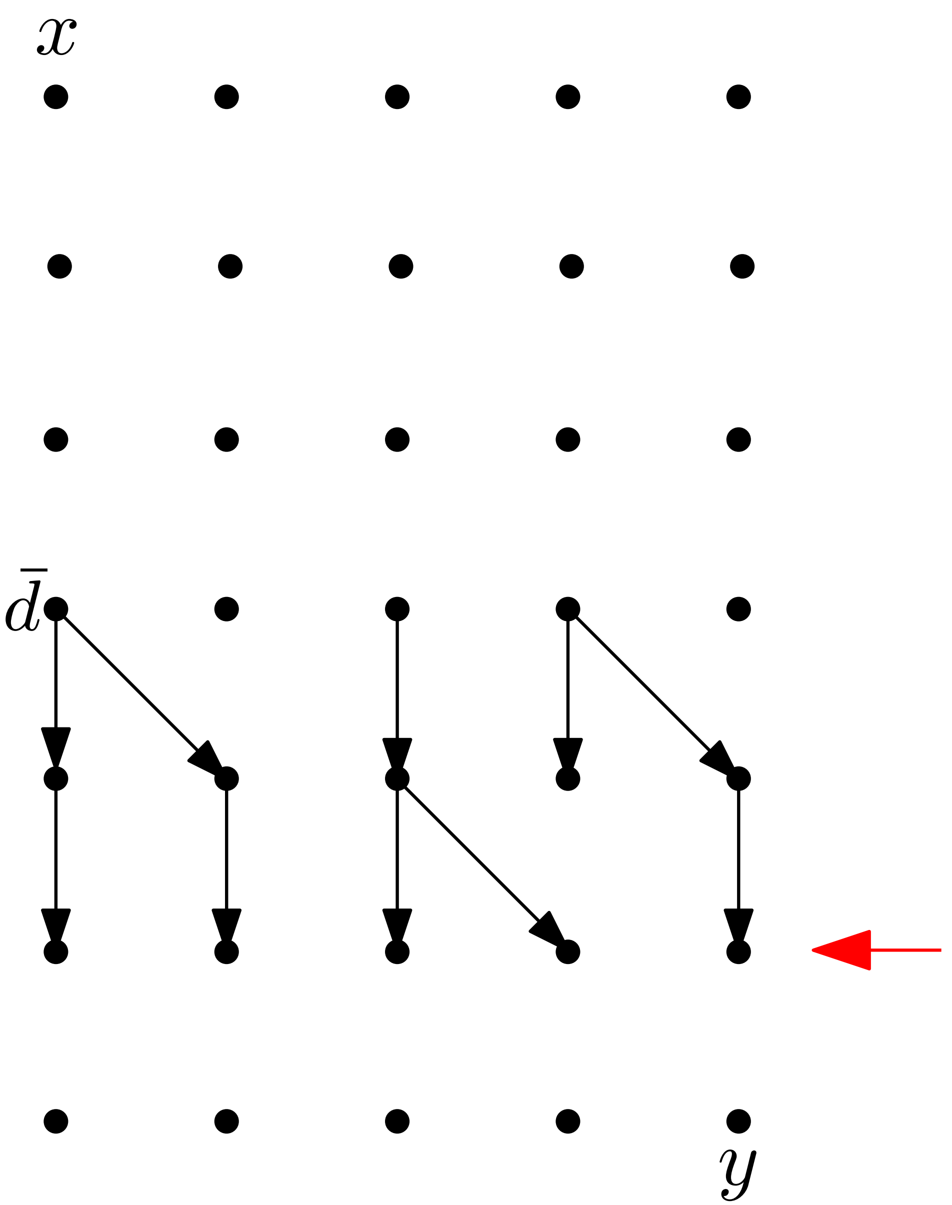

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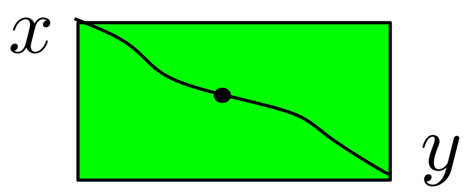

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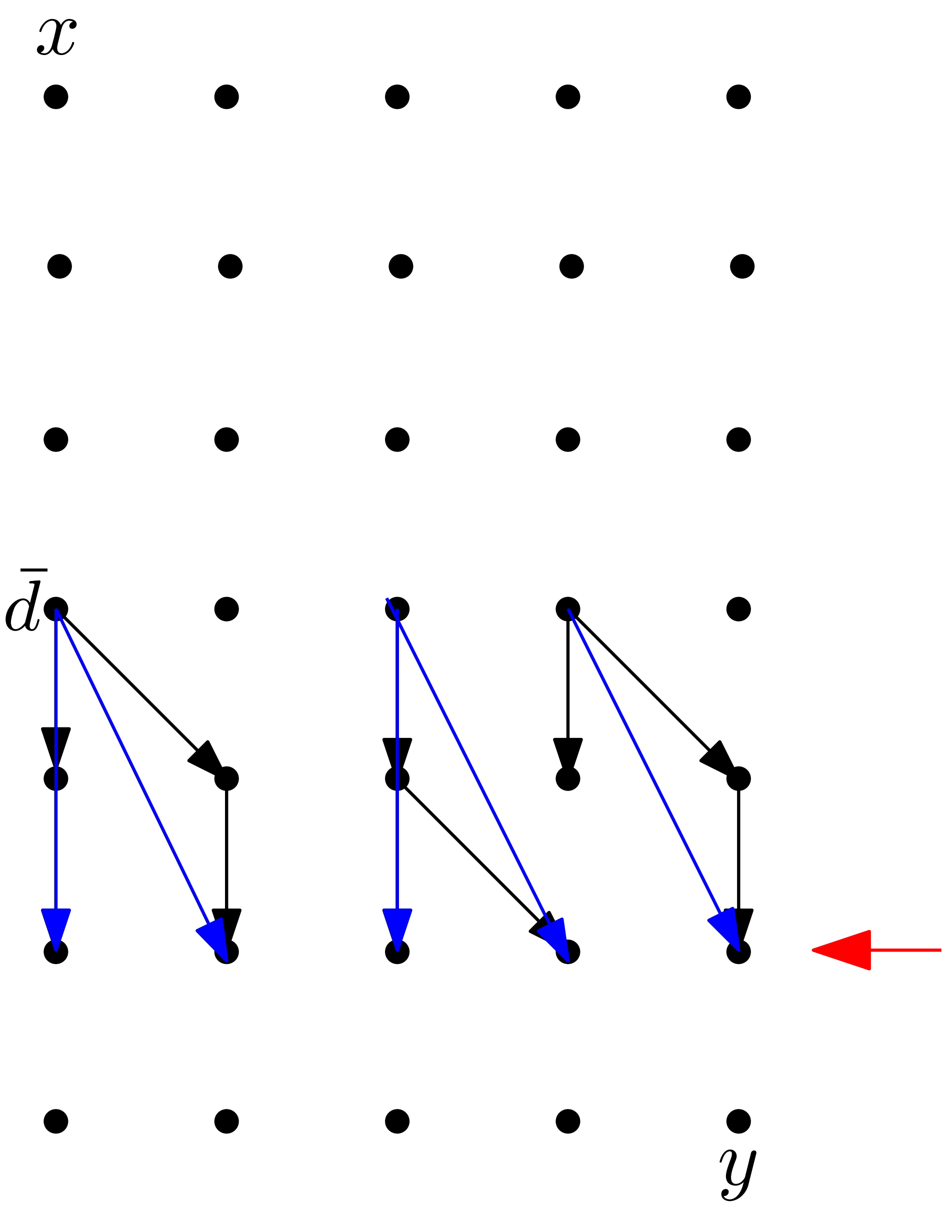

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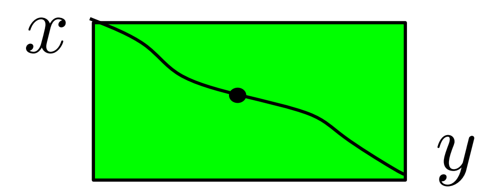

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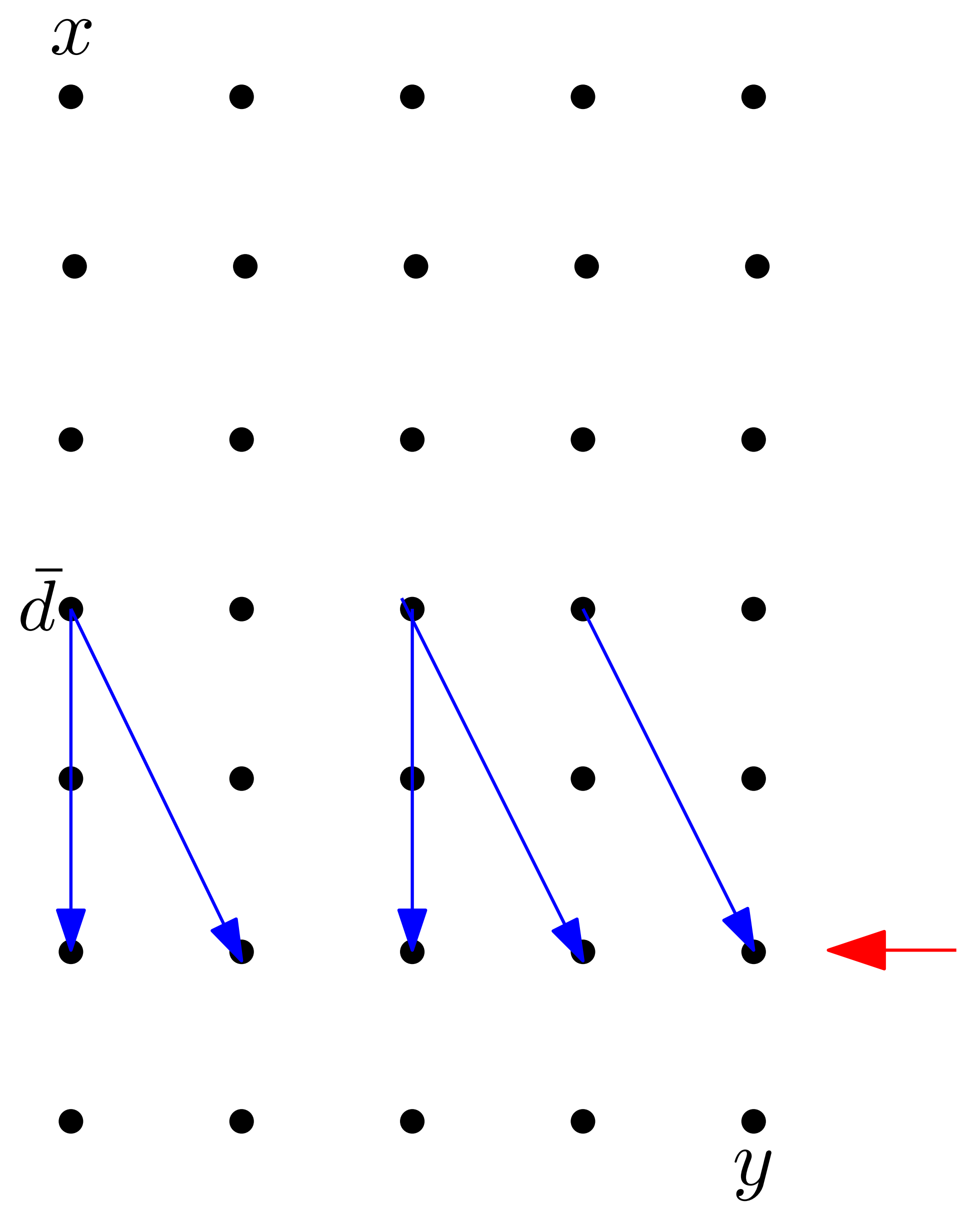

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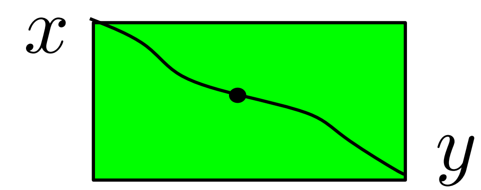

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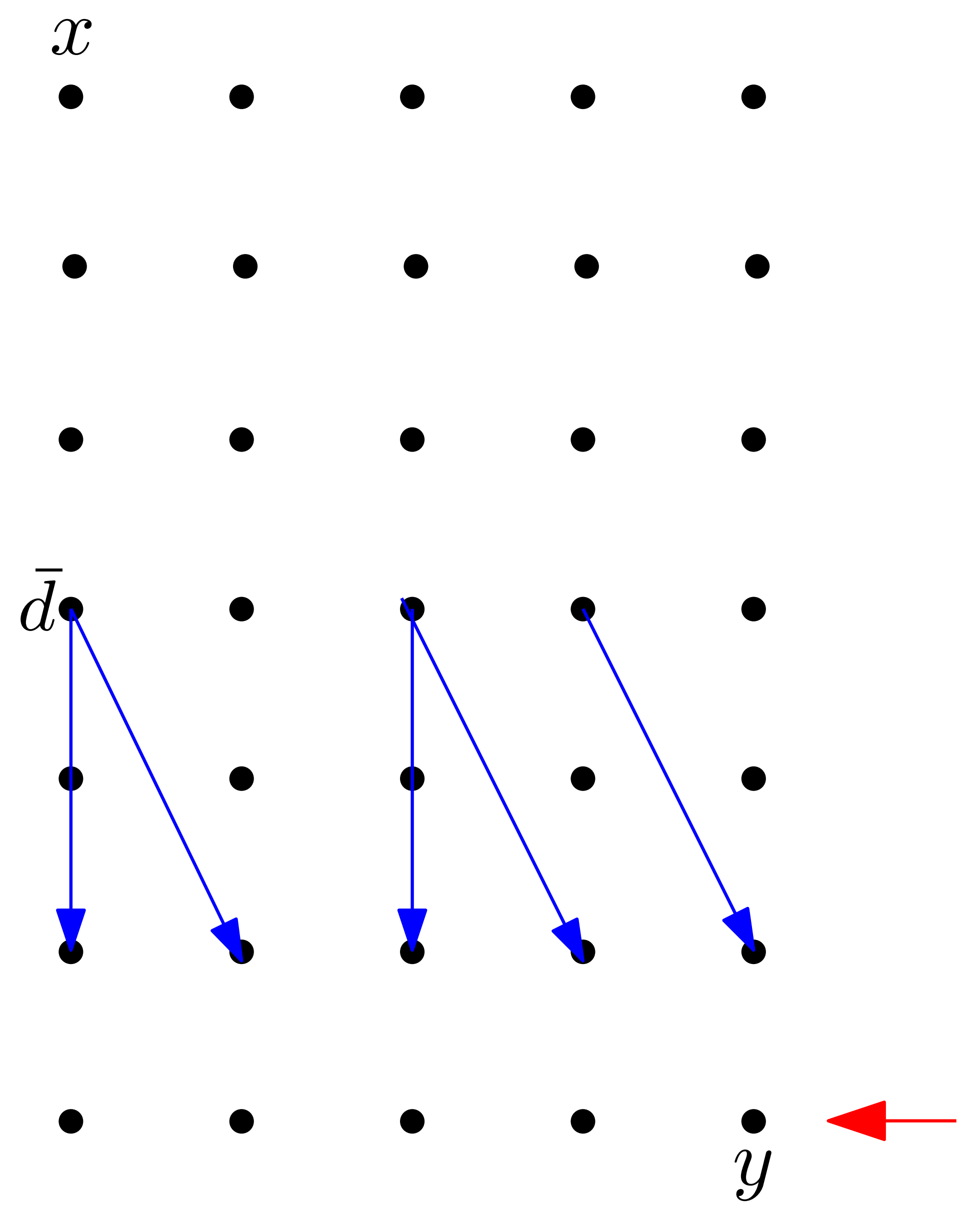

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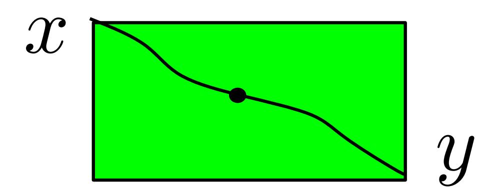

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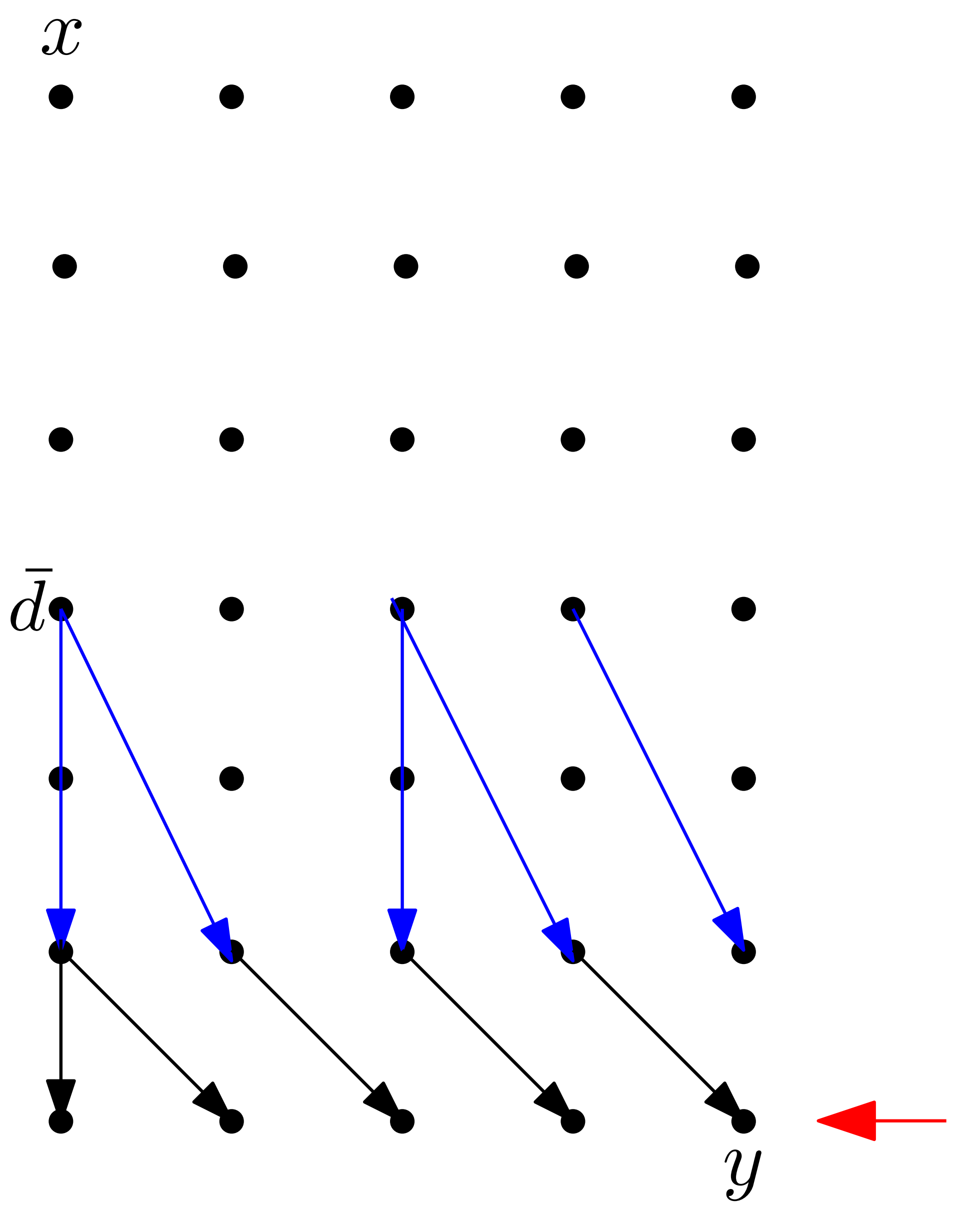
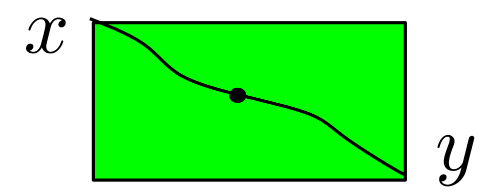
If $z_d = \bar{d}$, $P(z) = z$.
If $z_d > \bar{d}$, then $P(z) = P(pred(z))$ where $pred(z)$ is predecessor of z on min cost path.

All of the $C(z)$ and $P(z)$ on level d can be calculated in $O(y_d - x_d)$ time (Monge property) using only knowledge of $C(z')$ and $P(z')$ where z' on level $d-1$.



Implementing $Mid(x, y)$ in $O(D + n)$ space and $Area(x, y)$ time



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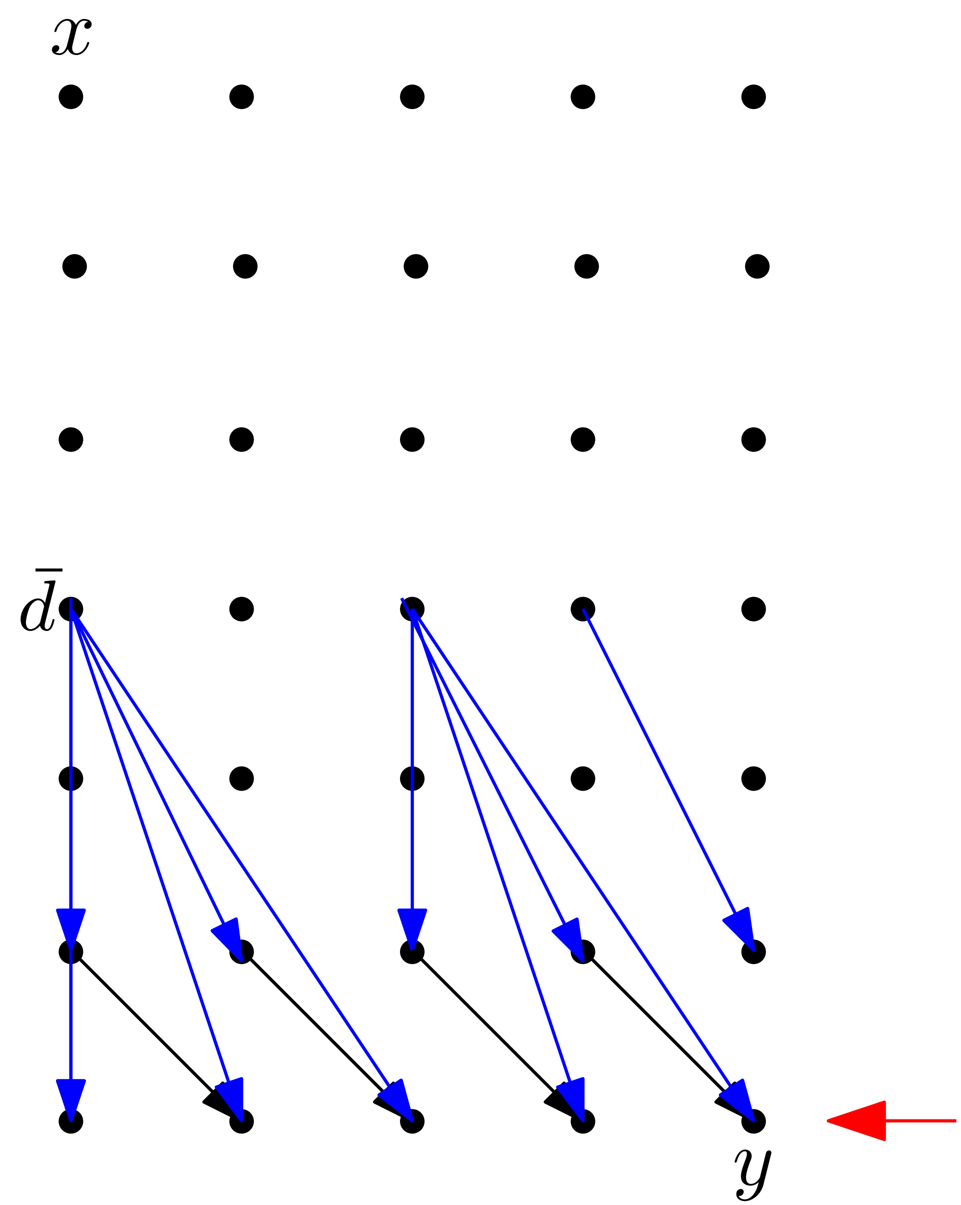

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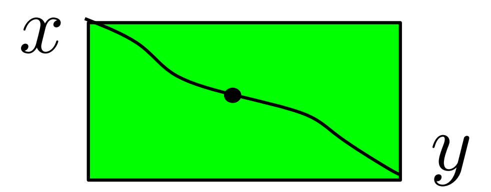

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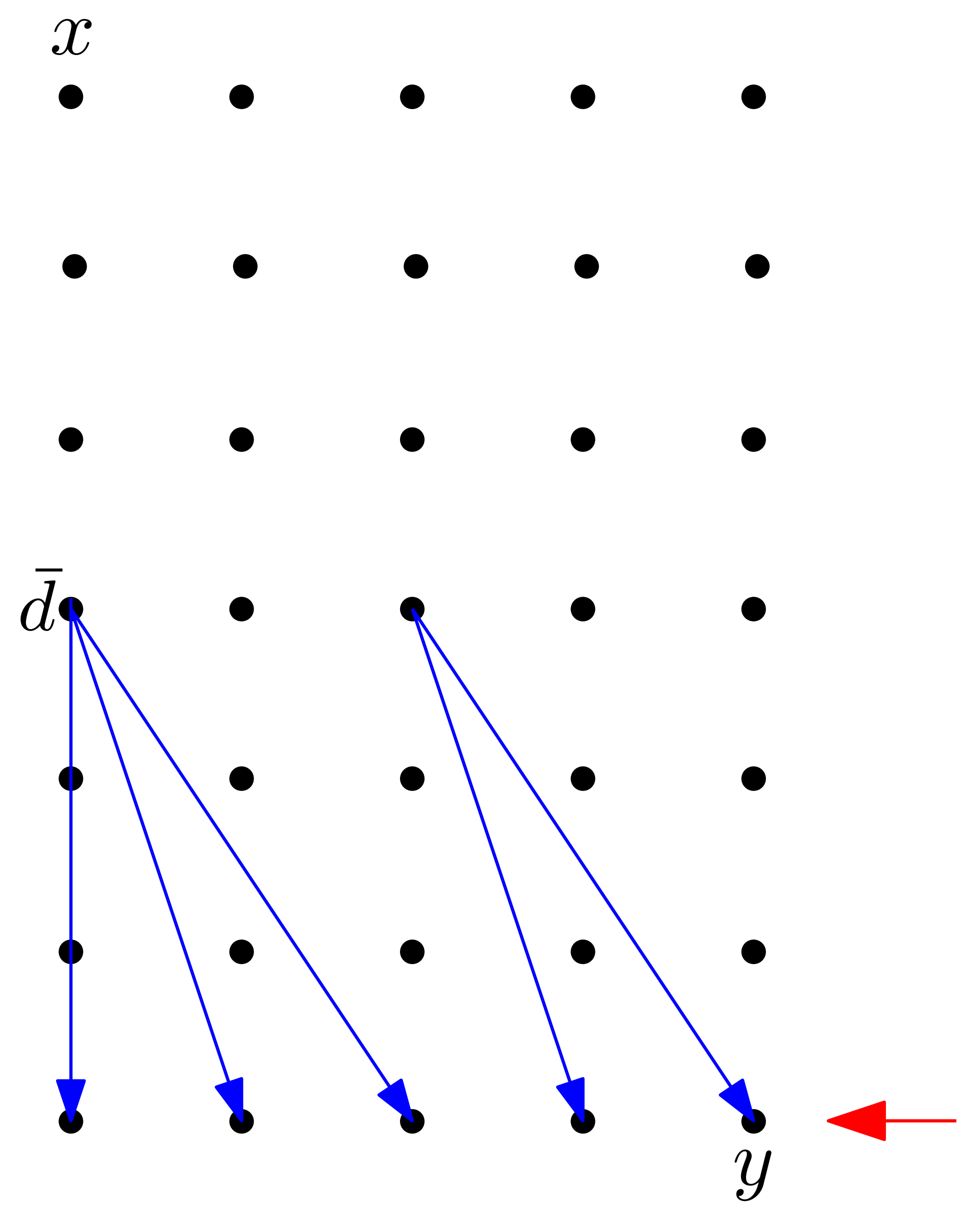

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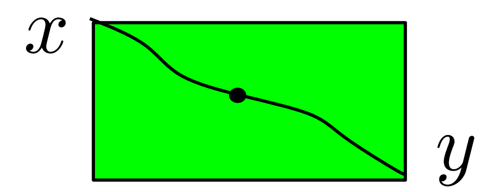

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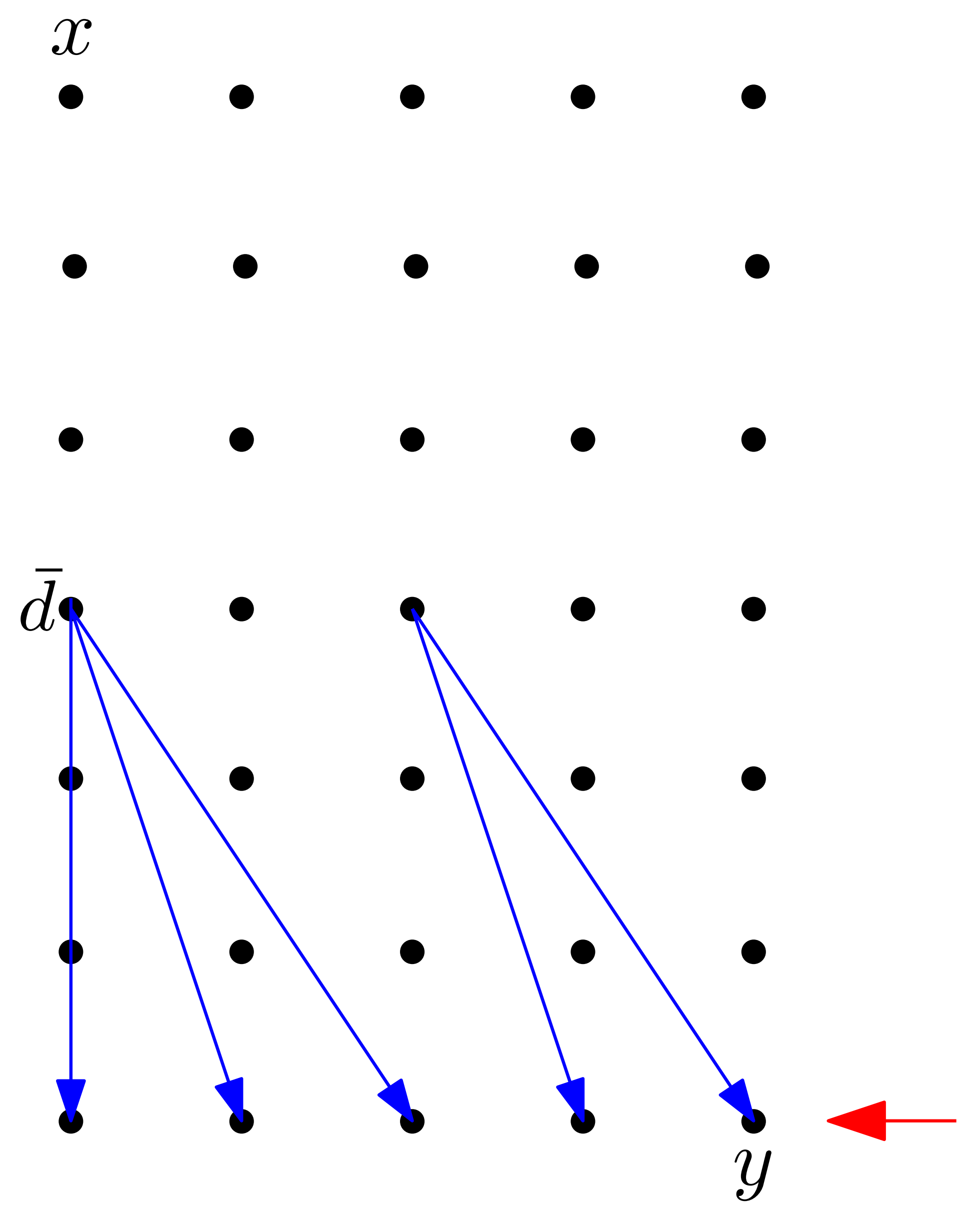

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$\Rightarrow$ can calculate value of $H(n, D)$ defined by

$$H(i, d) = \min_{0 \le j < i} \Big(H(j, d-1) + w(j, i) \Big) \qquad \begin{matrix} 0 \le i \le n \\ 0 \le d \le D \end{matrix}$$

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using $O(D+n)$ space and $O(Dn)$ time

Outline

- Review of the Monge Speedup
- Saving Space While Saving Time
- Conclusion

Conclusion

We just saw one technique for reducing time in dynamic programming and another for reducing space.

There are *many* such DP improvement techniques.

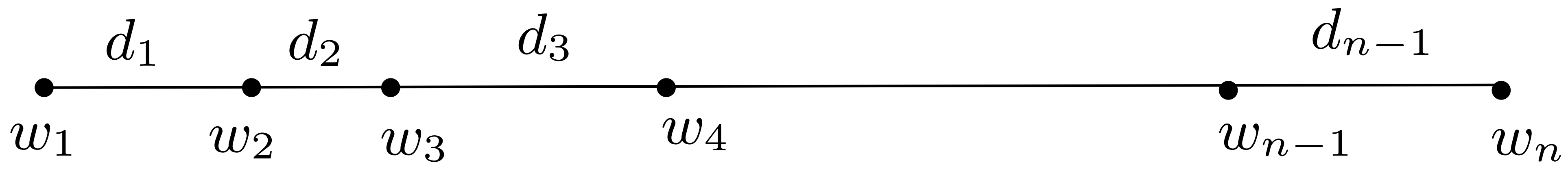
The problem is that they're they are all ad-hoc techniques, primarily known to specialists.

Need to develop a general theory of DP improvements, especially speedups, that is accessible to "users".

Goal is a recipe book that DP designers can check to see how to speed up their application-specific problems.

Open Question

- Two-Sided Online K-Median on a Line



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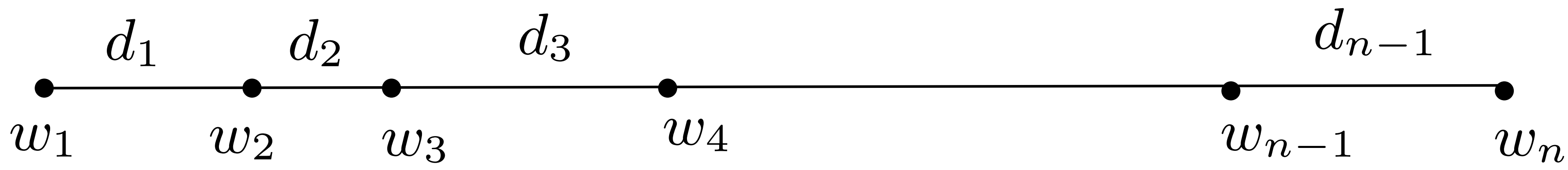

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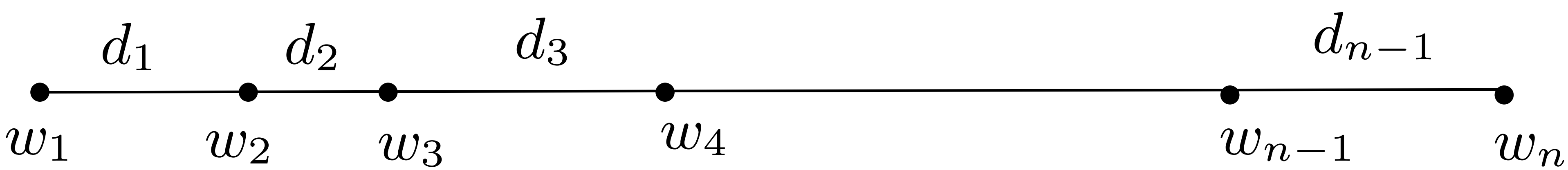

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Online Problem: Adding new elements to **right and left**.
Best known is $O(kn)$. Just as bad as reconstructing from scratch.
Is there a better way?