Approximate Path Problems in Anisotropic Regions

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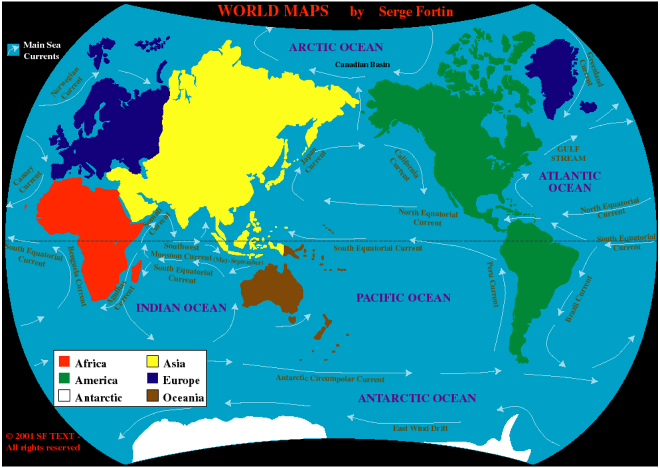
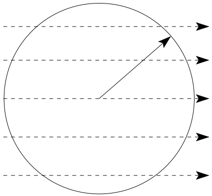
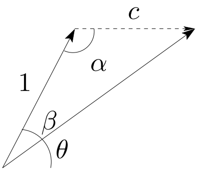
Joint work with J. Jin, H. Na, A. Vigneron, and Y. Wang

Paths in a Heterogenous Environment



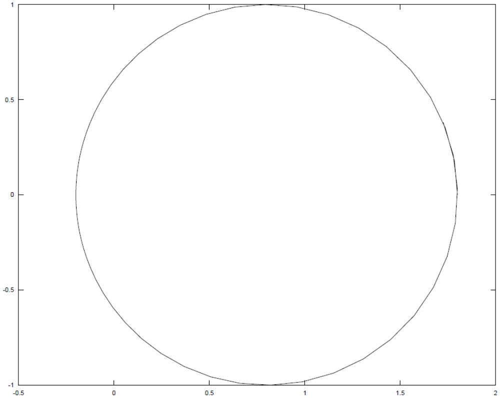
Paths in a Current





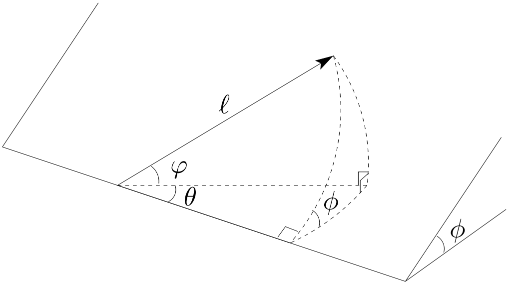
$$\begin{aligned}\text{speed} &= \sqrt{1 + c^2 - 2c\cos\alpha} \\ &= \sqrt{1 + c^2 + 2c\cos(\beta + \theta)} \\ &= \sqrt{1 + c^2 + 2c\cos(\arcsin(c\sin\theta) + \theta)}.\end{aligned}$$

Paths in a Current



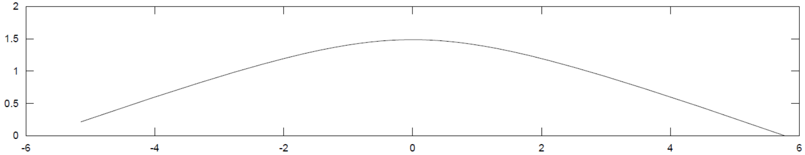
Unit disk

Paths on a Terrain



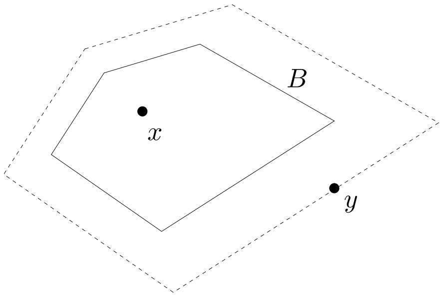
$$
\begin{aligned}
\ell &= \text{distance traveled,} \\
\mu &= \text{friction coefficient,} \\
\text{Energy} &= \ell(\mu \cos \phi + \sin \varphi) \\
&= \ell(\mu \cos \phi + \sin \theta \sin \phi).
\end{aligned}
$$

Paths in a Current



Unit halfdisk: $\phi = \pi/6$, $\mu = 0.2$.

Convex Distance Function



$$d_B(x, y) = \inf\{\lambda : y \in \lambda B + x\}$$

Non-negative, triangle inequality, possibly assymetric.

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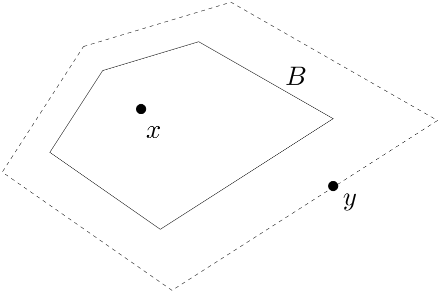


$$d_B(x, y) = \inf\{\lambda : y \in \lambda B + x\}$$

Non-negative, triangle inequality, possibly assymetric.

Assume that B is sandwiched between concentric disks of radii 1 and $1/\rho$.

Input

- A planar subdivision $\mathcal{T}$ possibly with some regions as obstacles.
- Assume triangular faces. Each face f is associated with a distance function d_f induced by a convex shape B_f.
- Given a path P in $\mathcal{T}$, we have

$$\mathrm{length}(P) \leq \mathrm{cost}(P) \leq \rho\,\mathrm{length}(P).$$

Previous Work

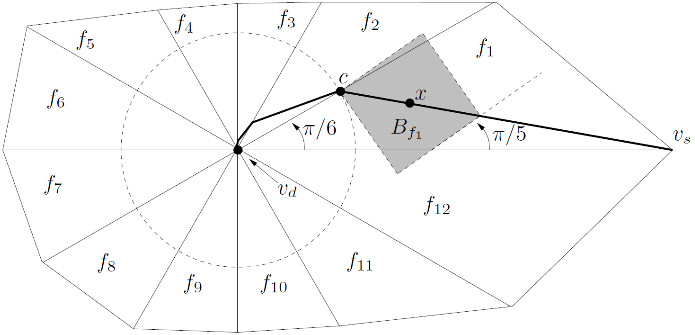
Weighted Region:

- Aleksandrov et al. [STOC00, JACM05]: dependent on the minimum angle in $\mathcal{T}$.
- Sun and Reif [Trans. Rob.05, Algo.06]: dependent on the minimum angle in $\mathcal{T}$.
- Mitchell and Papdimitriou [JACM91]
  $O(n^8 L)$ time: n is the number of vertices in $\mathcal{T}$, L is the number of bits in the input, which includes a term $\log(1/\epsilon)$.

Our results

- Approx. shortest path in $O(\frac{\rho^2 \log \rho}{\epsilon^2} n^3 \log \frac{\rho n}{\epsilon})$ time. [SICOMP08]
- Querying approx. shortest path [SICOMP10]:
  - query time $= O(\log \frac{\rho n}{\epsilon})$.
  - space $= O(\frac{\rho^2 n^4}{\epsilon^2} \log \frac{\rho n}{\epsilon})$.
- Approx. shortest homotopic path in $O(\frac{\rho^5 h^5}{\epsilon} k^2 n^3 \log^4 \frac{\rho k n}{\epsilon})$ time.

Infiniteness of the Optimal



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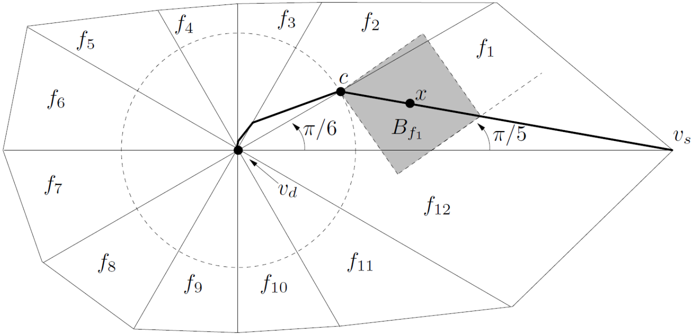
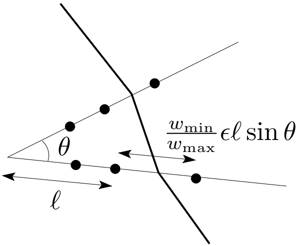


Existence of the shortest path can be proved using length spaces.

Theme of Previous Work



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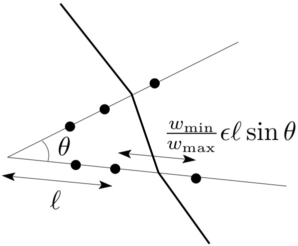

- Cost of link within the face $\geq w_{\min}\ell\sin\theta$.

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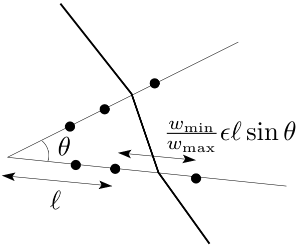

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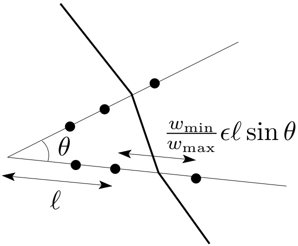

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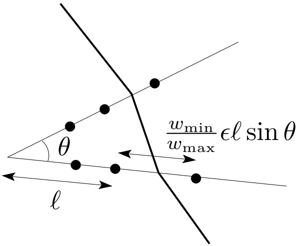

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- Overall relative error ϵ.

An Easy Lemma

Fix source v_s and destination v_d. Let n be the number of vertices in $\mathcal{T}$.

Focus on paths at most $k \geq 2n - 4$ links. Define path P_k^ϵ with at most k links such that

$\mathrm{cost}(P_k^\epsilon) \leq \left(1 + \frac{\epsilon}{3}\right) \cdot$ min cost of paths with at most k links.

Lemma

$\mathrm{cost}(P_k^\epsilon) \leq \frac{4\rho}{3}\mathrm{geo}(v_s, v_d)$.

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Proof. Let Q with a $\mathcal{T}$-respecting path with length $\text{geo}(v_s, v_d)$ with the minimum number of nodes. Q has at most $2n - 4$ links. Thus,

$$\text{cost}(P_k^\epsilon) \leq \left(1 + \frac{\epsilon}{3}\right)\text{cost}(Q) \leq \frac{4}{3}\text{cost}(Q) \leq \frac{4\rho}{3}\text{geo}(v_s, v_d).$$

A Simple Algorithm

1. Define the ball B_0 centered at v_s with radius $\frac{4\rho}{3}\mathrm{geo}(v_s, v_d)$. So $P_k^\epsilon \subset B_0$.

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2. For each edge e of $\mathcal{T}$, place a maximal set of Steiner points on $e \cap B_0$ with spacing $\frac{\epsilon}{6\rho k}\mathrm{geo}(v_s, v_d)$.

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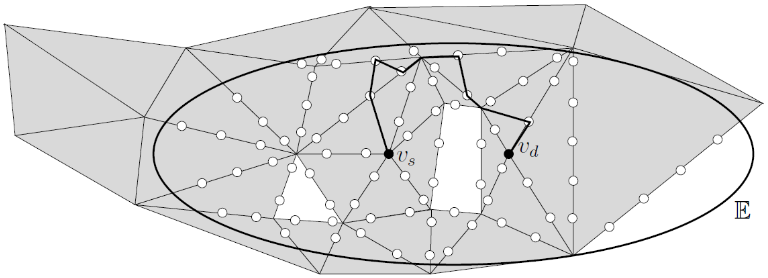

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3. Define a Steiner graph G:
   - Make a directed edge (p, q) for any Steiner points or vertices p and q of $\mathcal{T}$ that border the same face.
   - Define the weight of (p, q) as $\text{cost}(pq)$.
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Lemma

For any $k \geq 2n - 4$, we can approximate any path with at most k links in $O(nk^2\rho^4/\epsilon^2)$ time.

Further Improvements

- Use balls B_i of radii $\frac{4\rho}{2^i 3}\text{geo}(v_s, v_d)$ for $0 \leq i \leq \log \rho$. Let $B_{\log \rho + 1}$ be $\emptyset$.
- For each edge e of $\mathcal{T}$, discretize $e \cap (B_i \setminus B_{i+1})$ using spacing $\frac{\epsilon}{2^{i+1} 6k}\text{geo}(v_s, v_d)$.
- Use Sun and Reif's BUSHWHACK algorithm to avoid generating the edges of G.

Lemma

For any $k \geq 2n - 4$, we can approximate any path with at most k links in $O(\frac{nk\rho \log \rho}{\epsilon} \log \frac{k\rho}{\epsilon})$ time.

Path Complexity and Main Result

Lemma

For any $\epsilon \in (0,1)$, there is a $(1+\epsilon)$-approx. shortest polygonal path P with at most $21\rho n^2/\epsilon$ links.

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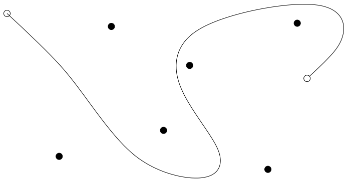
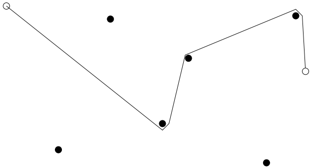
Approx. Shortest Homotopic Path





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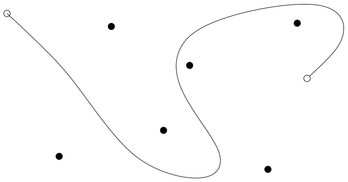
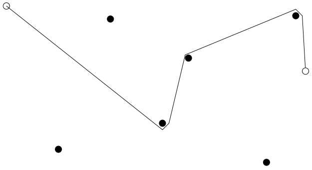
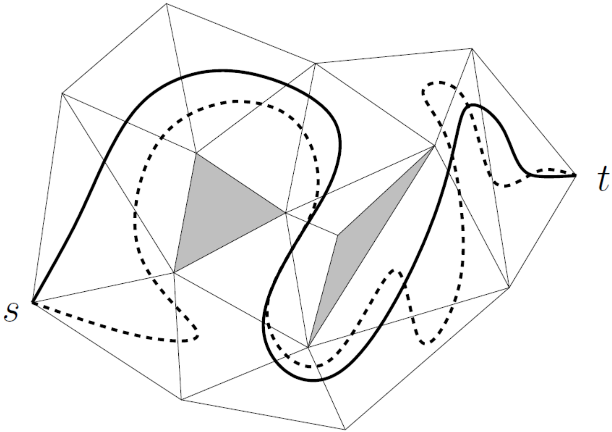




- Originate from VLSI research.
- Some planning system works by optimizing paths sketched by users.
- We need to require the convex distance functions to be symmetric.

Encoding the Homotopy



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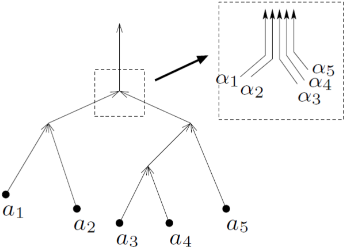
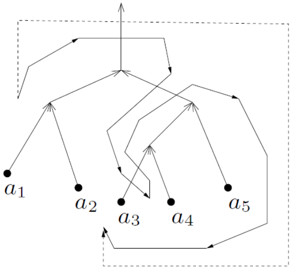
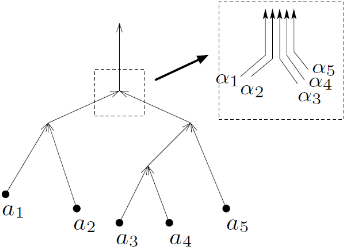
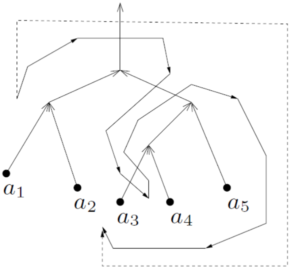




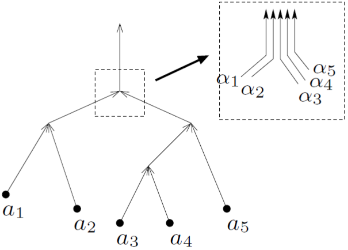
- Pick one vertex of each obstacle as an anchor.
- Compute an anchor tree: some approx. shortest path tree from the highest point in $\mathcal{T}$ to all anchors.

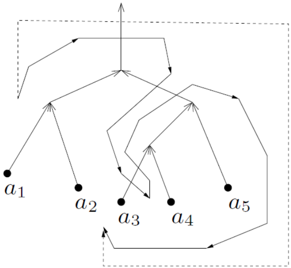




- Crossing sequence of the solid path: $\overrightarrow{a_1}\overrightarrow{a_2}\overrightarrow{a_3}\overrightarrow{a_4}\overrightarrow{a_5}\overleftarrow{a_5}\overleftarrow{a_4}\overleftarrow{a_3}\overrightarrow{a_3}\overleftarrow{a_3}\overrightarrow{a_3}\overrightarrow{a_4}\overrightarrow{a_5}$.
- It can be reduced to the canonical crossing sequence $\overrightarrow{a_1}\overrightarrow{a_2}\overrightarrow{a_3}\overrightarrow{a_4}\overrightarrow{a_5}$ of the dashed path.

Encoding the Homotopy





Lemma

Two paths from s to t are homotopic if and only if their canonical crossing sequences are identical.

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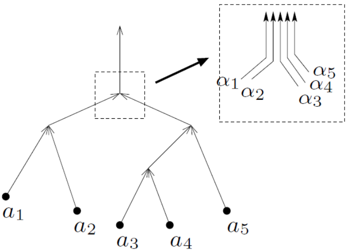
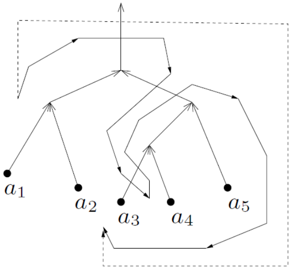
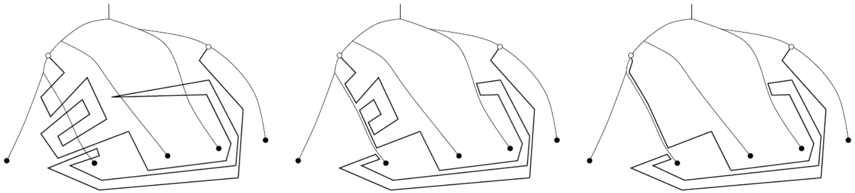




Lemma

For any ancestor-descendent points x and y in the anchor tree, the tree path cost between x and y is at most the shortest path cost between x and y plus $O(\epsilon^2)$.

High Level Strategy



High Level Strategy

1. Compute the canonical crossing sequence C of the input path.
2. Take some discretization $\mathcal{D}$ of the overlay of $\mathcal{T}$ and the anchor tree. Treat the anchor tree as an obstacle.
3. Compute shortest paths in $\mathcal{D}$ from s to all vertices of $\mathcal{D}$.
4. Let $\overrightarrow{a_i}$ be the first symbol in C. Let γ_i be the path in the anchor tree from a_i to the root. Copy the costs of reaching the vertices on left of γ_i to the right of γ_i.
5. Use the vertices of γ_i as multiple weighted sources and find shortest path to all vertices of $\mathcal{D}$ again.
6. Repeat last two steps until all symbols in C are processed.

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Lemma

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For any $\epsilon \in (0,1)$, *there is a* $(1+\epsilon)$*-approx. shortest polygonal path with* $O(\rho n^2 \log \frac{\rho n}{\epsilon})$ *links.*

Theorem

For any $\epsilon \in (0,1)$, *we can find a* $(1+\epsilon)$*-approx. shortest homotopic path in* $O(\frac{\rho^5 h^5}{\epsilon} k^2 n^3 \log^4 \frac{\rho k n}{\epsilon})$ *time.*

Summary

- Approx. shortest path in $O(\frac{\rho^2 \log \rho}{\epsilon^2} n^3 \log \frac{\rho n}{\epsilon})$ time. [SICOMP08]
- Querying approx. shortest path [SICOMP10]:
  - query time $= O(\log \frac{\rho n}{\epsilon})$.
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- Extend the cost model. For example, allow forbidden travel directions on a terrain.